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VIII. *Cohesion (Fifth Paper).* *By* HERBERT CHATLEY, *D.Sc. (Lond.)*.

RECEIVED NOVEMBER 3, 1918.

THE principal purpose of the previous Papers in this series has been to show that cohesion can be approximately represented by the excess of a central force located in the molecules over the kinetic repulsion between them, the said force varying according to a changing inverse power of the distance from centre to centre of the molecule pairs and becoming equal to gravity at the distance of a few molecular diameters.

Dr. Allen in a written communication criticising the fourth Paper expresses the opinion that "It is more than doubtful whether any attempt to found a theory on purely central attractions and repulsions varying as some power of the distance can prove adequate to explain the facts." He also refers favourably to Sutherland's hypothesis, and mentions that Lewis has shown that the Obach Walden relation between molecular pressure and the di-electric constant supports an electromagnetic basis for cohesion. In a previous communication (relating to the author's second Paper) he quotes Weiss as holding an adverse opinion on this magnetic hypothesis.

Jaeger's recent work on surface tension at various temperatures does not wholly confirm the Walden relation.

Sutherland's hypothesis appears (the author unfortunately cannot quote him at first hand) to be wholly electrostatic, and to postulate the attractive forces as due to the non-coincident arrangement of the mutually neutralising charges in a compound molecule, so that the molecule attracts another much in the same way as one of a pair of bar magnets. The writer has made some computations on this basis, and it appears that the force varies inversely as the fifth power of the distance when the molecules are very close, and as the fourth for moderate separation. At large distances the force is zero, and the hypothesis does not therefore explain gravitation. Sutherland has devised the hypothesis that the attraction between unlike charges exceeds the repulsion between like charges in the ratio one plus the reciprocal of 10 to the 43rd power. This means simply that the gravitational effect is superposed and does not appear to afford any explanation whatever of gravitation. The author is under the impression

—possibly erroneous—that cohesive force diminishes with separation more rapidly than is required by Sutherland's hypothesis.

Recent experiments by Svedberg in Sweden and Langmuir and Harkins in America appear to show that electromagnetic forces do play a considerable part in capillarity.

It may even be that the forces are both electrostatic and electromagnetic, the complexity of the vectorisation being explicable in this way.

However this may be, in the absence of a real knowledge of the mechanical structure of a molecule and its constituent atoms approximate formulæ for partially equivalent central forces cannot but be useful, especially if they can be related to the simple electrostatic bonds of nascent atoms on the one hand and the gravitational forces on the other.

For this reason the author adheres for the time being to the form

$$t_2=\frac{Gm^2}{d^{2+n/k}}$$

for the attraction. n being a little more than 4.

For the repulsion he would suggest the following form based on the standard expression for the "virial" or kinetic energy of a molecule and allowing for the actual molecular interstice $(k-1)d_0$,

$$t_1=\frac{RT}{Nd_0}\cdot\frac{1}{3(k-1)},$$

With further reference to the attraction formula, a consideration of all the data afforded by—

(*a*) Strength of materials,
(*b*) Tension of liquid films,
(*c*) Heats of fusion and evaporation,
(*d*) Internal pressure of gases, and
(*e*) Kinetic repulsion of gaseous molecules

points to molecular linkages of the order of 10^{-4} down to 10^{-6} dynes in the solid and liquid states. This then provides the initial values at moderate proximity, and it remains to discuss the rate of decrease with separation.

There is a very distinct relation between the rate of decrease and the ratio of the bond between an isolated molecular pair to the bond exerted by many molecules on one individual. If, for the purpose of hypothesis, we conceive the molecules to be spherical, with densest packing each will be surrounded

by 12 others in mutual contact (the rhombic dodecahedral arrangement) and the attraction exerted by a semi-spherical shell of molecules upon one central one in a direction normal to the diametral plane of the hemisphere will be as follows according to the law of diminution :—

Law of inverse variation.	Ratio of attraction to that of a single pair.
Square ..	Three per shell, integral for indefinite large mass is indefinitely large. This fact decisively controverts the inverse square hypothesis.
Cube....	Three for the first shell, $1\frac{1}{2}$ for the next, one for the third, and so on. Integral not convergent, but only increases slowly.
Fourth ..	Three for the first, three-quarters for the second, and so on, as the inverse square of the number. Integral to infinity equals six.
Fifth....	Three for the first, and then as the inverse cube. Integral to infinity equals four.
Sixth ...	Three for the first and then as the inverse fourth. Integral to infinity equals three and four-sevenths.

For the author's formula, with a varying power of the distance, the ratio is practically three—*i.e.*, the innermost shell only is effective.

For cubic packing, the ratios are practically halved (using semi-cubic shells) and for the author's formula only one molecule is effective.

The value of these ratios for real molecules which are certainly not spherical, is somewhat doubtful, but a maximum of three seems clearly indicated. Bragg's results with crystals indicate that the atoms and molecules are immovable in the solid, but a comparison of the diameters deduced from viscosity with those required to fill the volume shows that there is considerable open space. Probably the crystal is a kind of open framework with atoms in effective contact along various lines. The great changes in molecular volume which accompany chemical combination agree with this notion. The high conductivity and small heat capacity at low temperatures as well as the minute expansions from absolute zero to fluidity all seem to show that the atoms are practically in contact at certain points.

IX. *Notes on Lubrication.* *By* S. Skinner, *M.A.*

Received November 18, 1918.

I. Combined Effects of Viscosity and Compressibility in Lubrication.

How two oils of the same viscosity can have different powers of lubrication and why the vegetable and animal oils may under certain circumstances be better lubricants than the mineral oils require explanation. I think the explanation may be in the difference of compressibilities of the various oils. Generally speaking, the vegetable oils are less compressible than the mineral oils.

In my note book I find a short record of some measurements made in November, 1905, on the air pressures in the neighbour-

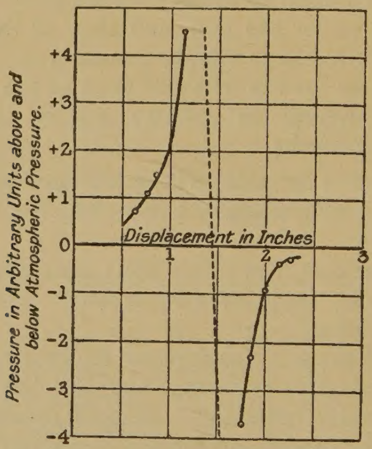
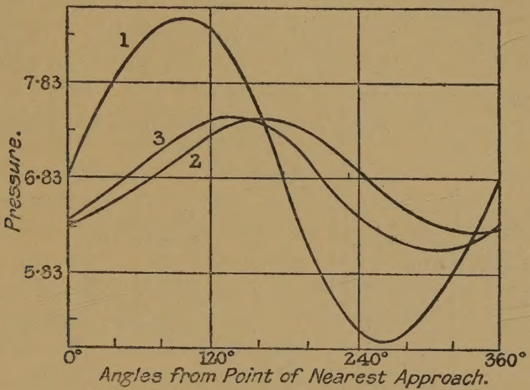


Fig. 1.

hood of a flywheel. A long, flat board had a small brass tube, ending flush with the under surface, passing through it. This was connected to a pressure gauge. The board was placed on the flywheel of a Willans and Robinson engine, which was revolving at 445 revs. per min. The board could be displaced, so that the opening of the brass tube passed over the point of contact of the board and flywheel. Observations were obtained on each side of the point of contact, and are shown in the diagram, Fig. 1. They exhibit the characteristics of lubrication with a compressible fluid.

In Harrison's Paper, Camb. Phil. "Trans.," 1913, an account is given of Kingsbury's experiments with air as a lubricant, and the theory for a compressible fluid is contrasted with that for an incompressible fluid. The following diagram, Fig. 2, is taken from this paper. It represents the pressure of



Speed, 1,730 revolutions per minute; $6·83 \times 10^4$ is atmospheric pressure measured in poundals per square foot.

FIG. 2.

air in the space between a complete cylindrical bearing and a journal.

Curve 1 is the theoretical curve for the air, if incompressible. Curve 3 represents Kingsbury's results, and Curve 2 represents Harrison's theory for a compressible fluid. Taking the positive portions of the curves, the areas are :—

Curve 1.	Curve 2.	Curve 3.
1,437	450	450

The bases of the curves :—

Curve 1.	Curve 2.	Curve 3.
56	50	50

∴ Mean pressures indicated by the curves :—

26	9	9

Comparison of the curves shows that for the incompressible fluid the maximum occurs earlier and the minimum earlier, and that the maximum and minimum are about $2\frac{1}{2}$ times as high as the experimental result. The property of compressibility prevents the reaching of high positive or negative pressures, and shifts the position of the maximum and minimum.

Suppose, now, that we have two lubricants of equal viscosity, but one less compressible than the other. The above remarks would lead us to expect that the first would be a superior lubricant to the second.

In the following table are collected the compressibilities of certain lubricants, and their viscosities. The table shows that the vegetable oils have great incompressibility; in fact, they approach water in this respect. I have not been able to find the corresponding data for animal oils, but I think it is likely that they are in the same order as those for vegetable oils.

—	Compressibility.		Viscosity.
Glycerine	$22 \cdot 1 \times 10^{-6}$	de Metz.	45 at 2·8°C.
Water	$47 \cdot 4 \times 10^{-6}$	,,	0·018 at 0°C.
Linseed oil	$51 \cdot 8 \times 10^{-6}$	,,	......
Almond oil	$53 \cdot 5 \times 10^{-6}$	,,	0·66 at 20°C.
Olive oil	$56 \cdot 3 \times 10^{-6}$	,,	0·81 at 20°C.
Castor oil	$47 \cdot 2 \times 10^{-6}$	,,	25·3 at 0°; 5·8 at 6·5°, 1·2 at 27°
Rape oil	$59 \cdot 6 \times 10^{-6}$	Quincke.	......
Paraffin	$62 \cdot 7 \times 10^{-6}$	de Metz.	......
C_6H_{14} (hexane)	$159 \cdot 0 \times 10^{-6}$	Bartoli.	......
$C_{16}H_{34}$	$75 \cdot 0 \times 10^{-6}$	,,	......
Vaseline oil ...	......	,,	0·91 at 20°

Dissolving a salt in water increases its resistance to compression. Mixing an incompressible with another more compressible fluid probably improves the lubricating power for certain purposes.

In conclusion, we may say that lubricity is a function of viscosity and compressibility. A good lubricant must possess high viscosity and high incompressibility. Is it not probable that the special property of "oiliness" claimed by technical experts in lubrication for certain oils is the physical property of incompressibility?

II. Cavitation in a Stretched Liquid.

In Worthington's Paper, on the mechanical stretching of liquids, "Trans." Roy. Soc., 1892, there is a note on a "Curious Phenomenon of Adhesion between two Solids immersed in a Stretched Liquid." Worthington says: "Desiring to ascertain whether an air free liquid would adhere under tension, as well to a metal as to glass, I enclosed a small piece of folded sheet copper in a glass bulb, which was then filled with boiled out air-free alcohol. Experiments with this showed that there was strong adhesion to the copper as well as to the glass, pro-

vided the vessel was kept still, but any agitation at once caused the stretched liquid to let go its hold at the point of contact of the copper and the glass. Close attention showed that the copper seemed to 'grow to the glass' at the point of contact, when the surrounding liquid was in a state of tension. This led to experiments on bulbs with smaller glass bulbs inside, and in all cases the loose bulb attached itself to the side of the vessel. The equilibrium was, however, very unstable. The release of the liquid took place on the slightest jar, the bubble always appearing at the contact of the solid with the wall, and the loose piece being generally tossed up when the rupture took place. I succeeded best with one small irregular bulb with a projecting stem; this could be gently waved about in the stretched liquid, while the foot of the stem adhered to a point on the side of the containing vessel (showing incidentally that considerable currents may exist in a stretched liquid)."

Worthington's explanation is based on the hydrostatic compression, due to the attraction of the two solids, on the thin layer of liquid between them. I think that the true explanation of the phenomenon may be found in the laws of lubrication. The small glass bulb is completely surrounded by the stretched liquid. If the bulb is against the side of the tube, and the tube is shaken in such a way that the bulb rolls, the pressure in the liquid near the point of nearest approach, and on the side towards which it rolls, will be increased, whilst on the side from which it rolls the pressure will be diminished. The conditions are similar to those of a cylinder rolling on a lubricated surface. The fall of pressure in such a case may be great enough to produce cavitation, and, if not sufficient, the break in the stretched liquid would naturally occur at this point, where the pressure is lower than the mean pressure. Worthington's experiment appears to me to be an evident illustration of the phenomena of lubrication occurring in a stretched liquid.

III. Delayed Boiling and Cavitation.

The phenomenon described in Section II. would naturally lead to consideration of other cases where somewhat similar conditions exist. One of these is to be found in the action of solid spheres in helping to prevent boiling by "bumping." When a liquid is heated in a very clean vessel it may be raised some degrees above the natural boiling point corresponding to the atmospheric pressure at the time. The liquid then

boils with explosive violence. Various methods are known to the chemist for preventing this bumping, which is a source of much annoyance. One is to introduce air by using fragments of porous earthenware, or small lengths of capillary tube closed at one end. Another method is to put into the flask beads which will roll about.

The action of the beads or marbles in rolling on the bottom of the vessel is similar to the rolling of ball bearings in oil-bath lubrication. When the balls roll in the oil they produce cavitation in the oil behind them. It is not necessary to explain the cause of the cavitating action. It is obvious that this is a case of lubrication.

The super-heated liquid is in an abnormally expanded condition like the liquid under tensile stress. To study this I heated some water in a clean beaker until the air was nearly all expelled, and the water reached the stage when large bubbles of steam are formed at a few points only. A thermometer in the water read 102°C., although the barometer was low on the day of the experiment. A slip of glass too long to go to the bottom of the beaker was placed at an angle across the beaker in the water. When this was rubbed with the round end of the thermometer bubbles of steam started in the wake of the point of rubbing contact. This is where cavitation would occur, and into any cavities thus formed the steam would enter, and owing to the high temperature it would be able to dilate the cavity into a large bubble.

Other experiments were made with a glass marble in a flask of alcoholic potash boiling over a water bath. Whenever the flask was tapped so as to move the marble a burst of vapour occurred on the side from which the marble moved. The same was observed with aqueous sodium hydrate solution. In Poynting and Thomson's "Heat" (Griffin & Co., 1904), page 165, it is stated that normal boiling is probably always associated with the presence of bubbles or cavities. An explanation of the action of bubbles of air is given. I think that from what has been said above the effect of cavities produced by cavitation between solids moving relatively to one another in the liquid should now be clear.

IV. Sounds Produced in Cavitating Liquids.

Air bursting into a vacuum produces sound. It might be expected that under suitable conditions sounds could be produced by allowing air to enter suddenly into the vacuum pro-

duced by cavitation. The phenomenon is not of importance, but seems to add interest.

A lens separated by a drop of oil from a flat glass plate is a suitable arrangement. If this is rocked quickly several times in succession, the oil works out to the side until only a narrow wall of oil separates the vacuum produced by cavitation from the air. Then further rocking is accompanied by the breaking of this wall by the rush of air into the vacuum. This is accompanied by a sharp click. It is surprising how loud the sound is for such a small cavity.

South Western Polytechnic Institute, Chelsea.
June, 1918.

ABSTRACT.

Experiments on the pressure of air in the neighbourhood of a flywheel running in contact with a flat tangential board are described to exhibit the properties of a compressible lubricant. A comparison of the compressibilities and viscosities of the vegetable and mineral oils leads to the conclusion that the special property of "oiliness" is the physical property of incompressibility. In Note II. Worthington's experiments on the adhesion of two solids immersed in a stretched liquid are explained as an illustration of the phenomena of lubrication in a stretched liquid. In Note III. the effect of glass beads, &c., in promoting the free boiling of air-free water is explained by the occurrence of cavitation behind the moving beads, &c., the steam entering the cavities thus produced and dilating them into large bubbles.

DISCUSSION.

Mr. T. C. Thomsen said that Principal Skinner's experiments were of great interest. It was a pity he had not been able to show his experiments with a glass bearing, in which the formation of cavity at the off side of the axle is clearly shown. The compressibility of fluids had been employed by Ferranti to convert air into a lubricant for very high speed spindles rotating at about 30 or 40 thousand per minute. At these high speeds the air is compressed, and a film is maintained between axle and bearing. He could not agree with the author's view that viscosity and compressibility were sufficient explanation oi oiliness. Actual bearings are not perfect, and if the lubricant fails it will do so at certain isolated high spots. These are regions of extreme pressure, and he thought the increase of viscosity which was known to take place at high pressures was important. A research is at present in progress on viscosities under high pressures, and if it is found that animal and vegetable oils have greater increases with pressure than mineral oils it will go a long way to explain the effects. There was another point about oiliness. It had been found that among mineral oils the best lubricants were those with a large proportion of unsaturated hydrocarbons. It is thought that the more of these that are present the more intimately the oil will adhere to a metallic surface. Now some of the animal and vegetable oils are very largely composed of unsaturated constituents, so that this property of adherence to metallic surfaces may readily be greater in these cases. He thought compressibility was of negligible importance. It should be borne in mind that the theoretical work which the author had quoted, and to which the diagram referred, only applied to perfect lubrication, with

nothing but fluid friction coming in. In practice, for this case, lubrication depended only on viscosity. It was when lubrication was only partial, with regions of metallic contact, that the property of oiliness came in. The question of cavitation could only come in at low and moderate pressures.

Capt. HYDE said he was interested in the question from the point of view of aircraft engines. Personally he did not think oiliness was a distinct physical property, but that it depended on the action of the oil on metallic surfaces. As Mr. Thomsen had pointed out, the theories of lubrication mentioned by the author referred to pure fluid friction, and there were no experiments to show that in these circumstances there was any difference between mineral and non-mineral oils of given viscosity. In 99 per cent. of cases there was metallic contact, and that was when oiliness came in. Experiments by Mr. Deely seemed to indicate that the differences lay in the action on the metallic surface. It did not seem to him that the difference between the compressibilities of, say, paraffin oil and rape oil would account for the difference in oiliness. Some experiments conducted at the National Physical Laboratory had revealed the curious fact that when working under severe conditions with imperfect lubrication there was, for the mineral oils a critical temperature of about 50°C. or 60°C., at which the lubrication broke down. No such temperature had yet been found with non-mineral oils.

Mr. C. R. DARLING did not think the experiments described by Principal Skinner on delayed boiling disproved the old explanation of the phenomena. There was always a film of air on glass which was very difficult to remove, and which was not removed by boiling. When actual rubbing takes place, as in the experiment mentioned, some of this may easily be liberated. He suggested trying the experiment with surfaces of quenched quartz.

Dr. BORNS, referring to the experiment in cavitation, asked whether Mr. Skinner had noticed a Paper by Tœpler, on the rupture of liquids between a rolling sphere and a plane plate, published in the "Annalen der Physik" last summer. Tœpler put a drop of liquid on a plain glass plate and placed a lens on the liquid. When the lens was rocked about, the liquid gave way, a series of crescents being formed. Tœpler examined these, and determined the critical velocity and stresses. A platinum wire would prevent delayed boiling. In this case the action must be rather on the lines suggested by Mr. Darling.

Mr. T. SMITH said that the difference in oiliness had been said to come in when the film had broken down. What happened then? Did the liquid reunite?

Capt. HYDE said he did not think it was known yet whether the better lubricant joined up quicker or not. In the case of pistons, he thought that even where you had "lubricated surfaces" there was never an actual oil film present. The lubrication was simply a condition of the surface.

Principal SKINNER said it was not clear how Capt. Hyde could be sure of his temperatures in the critical temperature experiments. In reply to Dr. Borns, he had seen Tœpler's Paper, but was reserving reference to it until a subsequent Paper, which he hoped to present to the Society.

X. *On Sir Thomas Wrightson's Theory of Hearing.* By W. B. Morton, *M.A., Queen's University, Belfast.*

Received November 22, 1918.

In his recently published book entitled "The analytical Mechanism of the Internal Ear," Sir Thos. Wrightson has put forward a new theory regarding the much debated question of the means by which the ear is enabled to separate out the constituent pitches in a compound aerial vibration. His theory has something in common with that which was proposed by Voigt with regard to the origin of combination tones,* inasmuch as in both theories attention is directed to the geometrical features of the curve which represents the motion or pressure of the air. In Voigt's theory the points of maximum and minimum displacement were shown to lie on certain sine curves having the frequencies of the combination tones, and this geometrical fact was supposed to give rise to sensations of the tones in question. Wrightson takes into consideration these stationary points along with the points where the curve crosses the axis. He groups all these two kinds of points together and calls them "Impulse Points," on the ground that there is "an impulse in the stapes and the liquid moved by it, not only where pressure and velocity become nil, which is synchronous with the crossing points of the pressure air-wave, but also at the crest of the wave where the acceleration of velocity and increase of pressure cease" (*loc. cit.*, p. 93). Having drawn a large number of curves got by compounding simple harmonic vibrations of different frequencies and having examined the spacing of their impulse-points, the author finds that there are in each complete cycle some points whose distance corresponds approximately to the frequencies of the component vibrations, others to the octaves below these pitches, others again to the summation and difference tones. These are all called "effective" points, and are held sufficient to account for the sensations of the corresponding tones. In all the special cases enumerated nearly all the impulse-points are stated to be "effective" in one way or another. The degree of approximation of the spacings to the correct interval is not stated.

* Cf., W. B. Morton and Mary Darragh, "On the Theories of Voigt and Everett Regarding the Origin of Combination Tones," in these "Proceedings," Vol. XXVII., p. 339, 1915.

I have not been able to follow clearly the argument by which the author seeks to establish the effectiveness of the impulse-points in relation to the mechanism of the ear. The purpose of the present note is, first, to indicate a method by which the distribution of the "impulse-points" can be examined without going through the labour of drawing each separate case, and second, to point out some difficulties in the way of accepting as valid the fundamental assumptions of the theory. Let the compound vibration be represented by

$$y=a\sin mx+b\sin n(x+\delta).$$

We shall suppose $m<n$, and shall call m and n the frequencies of the component vibrations. The points where the curve crosses the axis are given by

$$\sin mx/\sin n(x+\delta)=-b/a.$$

Accordingly if we plot the curve $y=\sin mx/\sin n(x+\delta)$ its intersections with the line $y=-b/a$ will give the positions of the points in question. We can thus, by altering the level of a horizontal line crossing the curve, follow the displacements of the zeroes of our compound vibration-curve, as the relative amplitudes of the components vary while a definite phase-relation, determined by δ, is maintained. It is easy to verify that, if the horizontal line is taken at levels both above and below the axis, we can exhaust all possible cases by letting δ run from o to $\frac{\pi}{2mn}$, in separate graphs.

The curve $y=\sin mx/\sin n(x+\delta)$ goes to infinity at the zeroes of the denominator. We can get a more compact graph by working only with amplitude-ratios lying between 0 and 1, so for cases in which the amplitude of the higher note n is the greater we plot the reciprocal $y=\sin n(x+\delta)/\sin mx$. The two curves can be combined in one diagram on which $y=$whichever of the two expressions is not greater than unity, the two parts of the graph being drawn with a full and a broken line respectively.

The stationary points on the vibration-curve are given by

$$ma\cos mx+nb\cos n(x+\delta)=0, \text{ or}$$

$$m\cos mx/n\cos n(x+\delta)=-\frac{b}{a},$$

and so can be discussed in a similar manner by plotting the graph $y=m\cos mn/n\cos n(x+\delta)$ or its reciprocal.

The accompanying figures show these graphs for the interval of a major sixth $m=3$, $n=5$, with $\delta=0$, $3°$ and $6°$, the curves for the two classes of impulse-points being combined on one diagram. The positions of the zeroes are given by the curves marked with a nought at the top and bottom, the curves giving the stationary points are marked with a cross.

Using these graphs we can trace as follows the displacements of the impulse-points as we pass from one note alone to the other alone, through all mixtures of the two, while the phase-relation remains constant.

The middle line of the diagram is crossed by the full-line curves at equi-distant points corresponding to the zeroes and crests of the lower note existing alone. If a horizontal line, starting from the central position is pushed upwards its points of intersection with the curves shift, showing the effect on the positions of the impulse-points of the addition of more and more of the higher note, the distance of the horizontal line from the axis being the ratio of the amplitude of the higher note to that of the lower. Fresh intersections appear when the moving line meets the loops of the graph and before the top of the diagram is reached (equal amplitudes) the number of points corresponds to the higher frequency. If now the line is brought down again its intersections with the *dotted* lines give the impulse-points when the higher note has the greater amplitude, the height above the axis now being the ratio of amplitude of lower to that of higher. Finally on reaching the axis again we have the spacing corresponding to the higher note alone. By carrying out the same movement on the lower half of the diagram we get the case in which one of the two component vibrations is reversed in phase.

It is possible to make certain general statements regarding the number of points of each class. The fresh points appear when the line is at a stationary value of y.

Considering first the zeroes, $y=\sin mx/\sin n(x+\delta)$ is stationary when $\tan mx/\tan n(x+\delta)=m/n$. It it easy to show that the corresponding value of y lies between m/n and unity, $(m<n)$. For

$$\frac{\sin mn}{\sin nx}=\sqrt{\frac{1+\cot^2 n(x+\delta)}{1+\cot^2 mx}}<1 \text{ since } \cot mx > \cot n(x+\delta)$$

and

$$\frac{\sin mx}{\sin nx}=\frac{\tan mx}{\tan n(x+\delta)}\sqrt{\frac{1+\tan^2 n(x+\delta)}{1+\tan^2 mx}}>\frac{\tan mx}{\tan n(x+\delta)}=\frac{m}{n}.$$

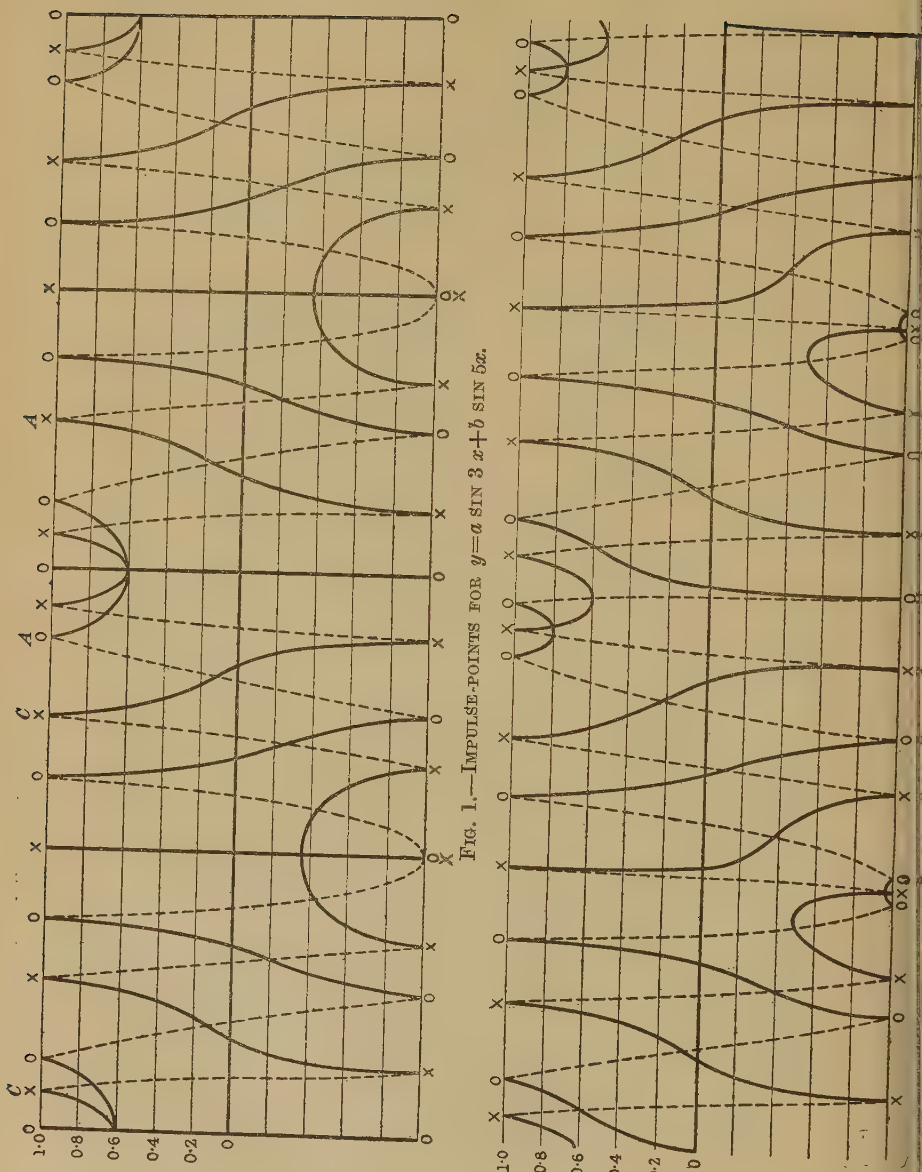


FIG. 1.—IMPULSE-POINTS FOR $y=a \sin 3x+b \sin 5x$.

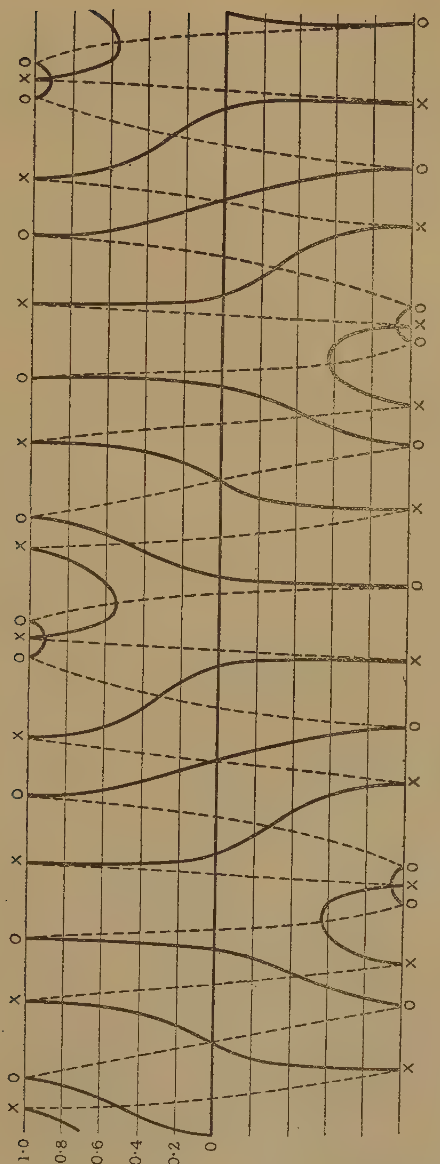


FIG. 3.—FOR $y=a \sin 3x+b \sin 5\,(x+6°)$.

It follows that when b/a, *i.e.*, amplitude of the higher note divided by that of the lower, is less than m/n, the number of zeroes is dictated by the lower note; if this ratio is greater than unity, by the upper note. For values of b/a lying between m/n and unity the number of zeroes on the compound vibration-curve depends on the phase relation-between the component vibrations.

The energies of the two components are as m^2a^2 to n^2b^2, and this may be taken as the ratio of the *physical* loudnesses of the notes. The rule just stated concerning the number of zeroes may then be stated as follows. The lower note dominates when it is louder than the upper, but the upper note does not certainly dominate unless its loudness exceeds that of the lower in a greater ratio than n^2 to m^2.

When the same method is applied to the maxima and minima of the curve, the critical values of b/a are found to be m^2/n^2 and m/n instead of m/n and unity. In terms of loudness this leads to a statement just the reverse of the former one; it is now the upper note which settles the number of points when it is the louder. The lower note does so, for all phase-relations, only when its loudness exceeds n^2/m^2 times that of the higher note.

Sir Thomas Wrightson's theory rests equally on two foundations, viz.: (1) That an impulse is communicated to the ear alike at the maxima and minima of the vibration-curve and at the points where the curve crosses the axis. (2) That the distribution of these impulse-points on the curve is such as to convey the sensations of the pitches which are heard. These two matters can be discussed independently of each other. I confine myself in this note to the second point, assuming for the moment that the supposed impulses actually occur. The following difficulties present themselves.

1. A point is classed as "effective" if it is preceded or followed (but not immediately) by another point at an interval corresponding to the frequency either of one of the two component pitches, or of the octave below either or of a combination-tone. The spacings of the combination-tones may be supposed to give rise to the sensations of those tones, but cannot help the ear to discern the two primaries. Further, there seems no sufficient reason for calling in the assistance of the lower octaves of the primaries. It is stated that these "assist the harmony" (p. 27), but no explanation is offered of the fact that they are not heard as pitches equally with the

primaries. If points reckoned effective by virtue of spacings of these two categories be ruled out, it makes a substantial reduction in the proportion of effective points.

But, independently of this, the fundamental difficulty is to see why the ear should pick out these particular spacings and ignore the others. With so many distances to choose from it is not surprising that a number can be found to approximate to any desired spacing. In order to make out a case for the special effectiveness of a spacing in impressing itself on the ear, it seems necessary to show that it occurs sufficiently often to raise it out of the mass.

I have gone into detail in one special case, the first one used by the author in explaining his theory (p. 26). This corresponds to the top line on my diagram (1), the interval of a major sixth when the notes have equal amplitudes and the two vibrations pass through zero at the same instant in opposite directions, *i.e.*, the case $y=a(\sin 3x-\sin 5x)$. We may call the notes C with period 10, and A with period 6. The complete cycle of the compound curve has period 30.

In taking a census of the spacings presented by this case I confine myself to distances under the half-cycle 15. Taking the 10 points marked with noughts and crosses (zeroes and stationary points respectively) on the left-hand half of the diagram, each of these is the left-hand limit of nine intervals, measured forward, which lie under the prescribed distance, giving $10\times 9=90$ spacings in all. These are made up of 39 different lengths, which are repeated, through the symmetry of the diagram; one occurs six times, six occur four times, 28 twice, and four once. Among those which occur twice are lengths 9·97 and 5·89. The former is a close approximation to C; the latter a worse approximation to A. These spacings are indicated by the letters above the diagram. There are two other lengths, each occurring twice, which lie at about two-thirds of a semitone on opposite sides of C, viz., 9·64 and 10·39. There is no other distance lying within a semitone of A. Thus, among the 90 intervals there are only six which could by any possibility suggest the pitch C and only two to suggest A. On the other hand, there are six exact intervals of 7·5, corresponding to the note F between C and A, which is not heard at all. There are 10 spacings lying within a semitone of the period 11.

It does not help much to bring in the lower octaves. There are two lengths of 12·01 and two of 12·10. To get approxima-

tions to 20 we should have to expend the range of spacings considered, and this would bring in a largely increased number of useless intervals.

2. Another consideration is the alteration brought about in the spacings of the impulse-points when the relative amplitudes are changed. Suppose, for example, that the ear has seized on the spacing marked *CC* in the case just considered, of equal amplitudes. The physical loudnesses of *C* and *A* are then as 9 to 25. Now suppose the loudness of *C* to be gradually increased to equality with *A*. The displacements of the impulse-points can be followed by bringing the horizontal line down to the level marked 0·6 and watching its intersections with the full curves. It will be seen that the two points in question separate to a distance 11·3, corresponding to a pitch below B flat, so this particular spacing would cease to be available long before equality of loudness is reached. The change is still more marked in the case of the points marked *AA*, where there is an approach equivalent to a rise of a fifth.

It may, of course, be urged that as certain pairs of points become gradually unavailable other pairs will become available. But it is hard to reconcile such an inexact and capricious means of determination with the certainty and constancy of the pitch-sensations to be explained.

A similar objection might be based on the alterations in the spacings which are brought about by changing the phase-relation with constant loudness-ratio. This indeed would be a more cogent argument because such progressive changes in phase-relation must always occur except when the tuning is perfect. In any case the actual relation between the phases is a matter of chance and has no influence on the discrimination of the pitches by the ear.

3. The simple harmonic air-motion which gives the sensation of a single pure tone has four impulse-points in each period, viz., two zeroes and two stationary points. In other words if the frequency of the note is n the impulses, according to the theory, fall on the ear with frequency $4n$. One would expect then that also in the case of a compound disturbance the sensation of pitch n would depend on the recurrence of impulses with frequency $4n$, whereas it is sought to be explained by spacings of frequency n.

4. The author adduces as an analogy the case of Seebeck's siren, in which a current of air is directed against a row of

holes spaced round the circumference of a rotating plate. I would suggest that a rough test of his theory might be made if the holes were given the relative positions of the consecutive impulse-points on a compound vibration-curve. This would be analogous to the plan adopted by Koenig when he sought to establish, by his "wave-siren," his views as to the effect of phase-relation upon quality of tone. It would be open to criticisms similar to those urged by Helmholtz against Koenig, but if the arrangement succeeded in giving clearly the two primary pitches this would be recognised as giving support to Sir Thomas Wrightson's views.

The theory has already attracted a considerable amount of attention, specially from those who are dissatisfied with the Helmholtz theory of the mechanism of hearing. Every attempt to discover a fresh point of view in so difficult a matter is to be welcomed.

The book contains a valuable and clear account of the internal anatomy of the ear, in which Prof. Keith has collaborated. Such discussions as I have seen have been directed to the internal processes and have not touched on those preliminary assumptions to which the criticism of the present note is directed.

ABSTRACT.

The theory seeks to explain the power possessed by the ear of analysing into its component tones a compound aerial disturbance. It assumes that (1) impulses act on the mechanism of the ear corresponding to the maxima and minima of the compound vibration-curve, and also to the points where the curve crosses the axis; (2) that among the spacings of these impulse-points there is a preponderance of intervals which approximate to the periods of the component tones, their lower octaves and their combination tones, and that these spacings determine the sensations of the component tones. The present note is concerned with the second of these assumptions. Graphs are drawn which exhibit the way in which the distribution of impulse-points varies when relative intensities and phase-relation of the component notes are changed. Difficulties are found in (1) the large number of other spacings presented to the ear, (2) the variations of the spacings with loudness-ratio and phase relation, (3) the fact that in a single pure tone the spacing is a quarter of the period of the vibration.

DISCUSSION.

Prof. Bayliss said he would like to make a mild protest against Prof. Morton's statement that physiologists welcomed Wrightson's Theory. It is really the anatomist who says that there is no structure in the ear capable of acting as a resonator, ignoring Helmholtz's demonstration that a membrane stretched unequally in different directions will resonate.

Personally he saw no objection to the resonance theory. On Sir Thos. Wrightson's theory there were four impulse points per vibration, If you reduce the frequency of vibration far enough you cease to hear a sound and hear the separate vibrations. Why do you not notice four impulses per vibration in this case. Another important point is one that Lord Rayleigh raised. We can hear sounds up to 30,000 per second. Now from what we know of nerve fibres, they are completely inactive for at least a thousandth of a second after receiving an impulse so they cannot possibly record more than a thousand separate impulses per sceond. Thus, on Wrightson's theory, all sounds requiring more than 1,000 impulse points per second should sound alike.

Principal Skinner was interested to hear Mr. Bayliss's opinion that the material of which the body is made is not acoustically the best. He had always had a difficulty in understanding how the cavity of the mouth could act as a resonator at all; the vibrations should be damped down so quickly.

Prof. Bragg asked if it was not the case that a sound was recognisable if two or three consecutive vibrations were present. This would explain Mr. Skinner's difficulty, for even if the vibrations of speech were very rapidy damped out, there might still be sufficient of them to give the note its characteristics.

XI. *Electrical Theorems in Connection with Parallel Cylindrical Conductors.* By ALEXANDER RUSSELL, *M.A.*, *D.Sc.*

RECEIVED NOVEMBER 26, 1918.

THE electrostatic problem of two conducting spheres having given electric charges and surrounded by a uniform dielectric has been completely solved. The capacity and potential coefficients of the spheres, the density of the surface charges, the potential at any point of the field and the mutual force between them can be computed in all cases without difficulty, to any required degree of accuracy.*

Unfortunately the apparently much simpler problem of two parallel cylindrical conductors has not been completely solved. It will be helpful, therefore, to give simplified proofs of the solutions already obtained and show how they can be extended. It is customary to make the assumptions that the cylinders are infinitely long, and that the other conductors of the system are at a great distance away from them. In this case it is shown that the three capacity coefficients are connected by two simple relations which determine the limits between which they must lie. In certain cases also their approximate values can be found. It is shown how the solution of the electrostatic problem enables us to solve the analogous problem of two parallel cylindrical conductors carrying high-frequency currents. In certain cases the exact values can be obtained of the current density on the surface, the inductance coefficients and the force between the conductors. In other cases useful approximations are given.

The Electric Force at any Point due to Two Parallel Cylindrical Conductors having Equal and Opposite Charges of Electricity.

Let us first consider the case of two thin parallel wires cutting the plane of the paper perpendicularly at A and B (Fig. 1). We shall suppose that they are infinitely long, that they have charges $+q$ and $-q$ per unit length respectively, and that the dielectric is air. To find the force at any point P in the plane of the paper join AP and BP and make the angle BPC equal to the angle BAP. Then PC will be the direction of the resultant force and its magnitude will equal $4qr/(r_1r_2)$ where $AB=2r$, $AP=r_1$, and $BP=r_2$. To prove

* Russell's "Alternating Currents," Vol. I., Ch. VIII., 2nd Ed.

this, we notice that the component force PQ due to the charge on the A wire$=2q/r_1$, the component force PS due to the charge on the B wire $=2q/r_2$, and the angle PSR=the angle APB. Hence the triangles APB and PSR are similar, and so the angle BPC equals the angle PAB. Also since $PR/PS=AB/r_1$, PR which is the resultant force, F at P, is given by

$$F=\frac{4qr}{r_1 r_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The potential v at the point P is the sum of the potentials due to the charges on the two wires, and hence

$$v=2q\log r_2-2q\log r_1=2q\log(r_2/r_1). \quad . \quad . \quad (2)$$

Since the angle BPC equals the angle BAP, the tangent PC to this circle at P gives the direction of the resultant force. Hence the tangent at every point of this circle and

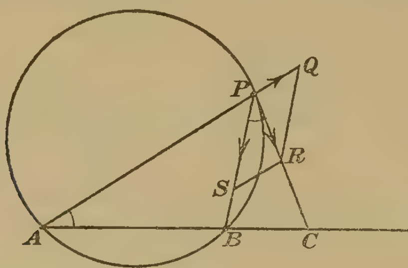


FIG. 1.

PR is the resultant force at any point P. The angle BPC equals the angle BAP. Hence every circle through A and B is a line of force.

therefore also at every point of any circle through A and B will give the direction of the resultant force. It follows that every circle which passes through A and B is a line of force. Again, if with centre at a point C on AB produced and with radius equal to $(CA\,.\,CB)^{\frac{1}{2}}$ we describe a circle, any radius CP of this circle will be tangential to the circular line of force through ABP, and hence this circle will cut all the lines of force at right angles. It is, therefore, the cross-section of an equipotential surface. We see that all the equipotential surfaces round A and B are cylindrical in shape, that their axes are parallel and that the centres of their cross-sections lie on AB or BA produced. The equipotential surfaces surrounding A, we shall call the A cylinders, and those surrounding B the B cylinders. It is to be noticed that if we

take any A cylinder and any B cylinder, A and B are the inverse points of their circular cross-sections.

By Green's theorem we can suppose that any A cylinder and any B cylinder become conductors without affecting the distribution of the flux external to them. Similarly, if two of the A cylindrical surfaces become conducting the distribution of the flux between them will not be affected. The surface density σ also at any point on the surface of these conducting cylinders is given by $F/(4\pi)$, where F is the electric force at the point. Hence the surface density at the point P of the equipotential surface is given by

$$\sigma=\frac{qr}{\pi r_1 r_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let the circles in Fig. 2 represent the sections of an A and a B cylinder respectively, and let their radii be a and b. Then, since A and B are the inverse points of the two circles,

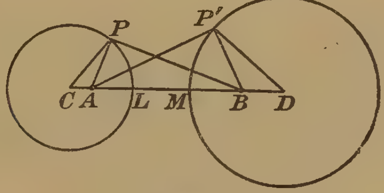


FIG. 2.
A and B are the inverse points of the two circles.

$CA \,.\, CB=a^2$ and $DB \,.\, DA=b^2$, where C and D are the centres of the two circles. The circle described on AB as diameter is the section of the smallest equipotential surface. It cuts the cylinders at right angles. If $2r$ be the diameter of this circle and if we denote the distance CD between the axes of the cylinders by c, we have

$$c^2r^2=4s(s-a)(s-b)(c-s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\qquad 2s=a+b+c$, (l.c., *ante* p. 164).

Since $r_2/r_1=BL/AL$, it is easy to show that

$$r_2/r_1=BC/a=a/CA.$$

Hence, by (2), $\quad v_1=2q \log (BC/a)=2q\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

where v_1 is the potential of the A cylinder and $\alpha=\log (BC/a)$.

Hence, $\varepsilon^{\alpha}=BC/a$ and $\varepsilon^{-\alpha}=CA/a$,

and thus $2r=CB-CA=a(\varepsilon^{\alpha}-\varepsilon^{-\alpha})$,

and so $r=a \sinh \alpha$.

Hence,

$$\alpha=\sinh^{-1}(r/a)=\log_{\epsilon}\{r/a+(1+r^2/a^2)^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (6)$$

It will be seen that α can be readily computed by (4) and (6).

We see from (5) that whatever the radius of the B cylinder may be, we have

$$q/v_1=1/(2\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Similarly, if v_2 be the potential of the B cylinder whose radius is b, we have

$$q/v_2=-1/(2\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since all the B cylinders have a charge of $-q$ per unit length. The value of β is given by

$$\beta=\sinh^{-1}(r/b)=\log_{\epsilon}\{r/b+(1+r^2/b^2)^{\frac{1}{2}}\} \quad . \quad . \quad . \quad (9)$$

We deduce from (7) and (8) that

$$\frac{q}{v_1-v_2}=\frac{1}{2(\alpha+\beta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is the formula for the capacity between the two cylinders—the capacity usually wanted in practice. Formulæ (7) and (8), however, are useful and instructive.

If the angle PCA in Fig. 2 by denoted be θ, we find by (3) that the surface density σ at P is given by

$$\sigma=\frac{qr}{\pi r_1 r_2}=\frac{q}{2\pi a}\cdot\frac{\sinh \alpha}{\cosh \alpha-\cos \theta}. \quad . \quad . \quad . \quad (11)$$

Similarly, the surface density at any point P' on the B cylinder will be given by

$$\sigma=-\frac{q}{2\pi b}\cdot\frac{\sinh \beta}{\cosh \beta-\cos \varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where φ is the angle $P'DA$ (Fig. 2).

If we write $\omega=\alpha+\beta$, we have

$$q/(v_1-v_2)=1/(2\omega), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$\cosh \omega=(c^2-a^2-b^2)/(2ab). \quad . \quad . \quad (14)$$

Let us now consider the case of a cylinder inside a hollow conducting cylinder, the axes of the cylinders being parallel but not necessarily coincident. Let the radius of the inner cylinder be a, the inner radius of the outer cylinder be b,

and let c be the distance between their axes. Then if q be the charge per unit length on the inner cylinder, $-q$ will be the induced charge per unit length on the inner side of the outer cylinder. The electric field between them, and therefore the potential difference between them, will be identically the same as that between two A cylinders whose radii are a and b respectively, the distance between their axes being c. The potentials v_1 and v_2 of these two cylinders when one of the B cylinders has a charge $-q$ per unit length, will be given by

$$v_1=2q\alpha \quad \text{and} \quad v_2=2q\beta\,;$$

and hence

$$\frac{q}{v_1-v_2}=\frac{1}{2(\alpha-\beta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This equation therefore gives us the capacity between the inner and the outer cylinder. If we denote $\alpha-\beta$ by ω_1 we easily find that

$$\cosh\omega_1=(a^2+b^2-c^2)/(2ab). \quad . \quad . \quad . \quad . \quad (16)$$

The surface densities at points on the surface of the inner cylinder, and on the inner surface of the outer cylinder, are given by (11) and (12) respectively.

Since q is the charge per unit length of the cylinder, the radius of which is a, it follows at once from (11) that

$$\int_0^\pi \frac{\partial\theta}{\cosh\alpha-\cos\theta}=\frac{\pi}{\sinh\alpha}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This equation can easily be verified by the calculus.

The centroid line of the distribution of the electrical charge on the cylinder whose radius is a (Fig. 2) will from symmetry lie in the plane passing through the axes of the two cylinders. If $\bar{x}$ be its distance from O, we have

$$\bar{x}q=2\int_0^\pi \sigma a^2\cos\theta\,\partial\theta=qa\varepsilon^{-\alpha}, \text{ by (17)},$$

and thus

$$\bar{x}=a\varepsilon^{-\alpha}=OA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Similarly, the centroid line of the charge on the B cylinder will pass through B.

It will be seen, therefore, that the inverse lines of the cylinders which pass through A and B respectively are the centroid lines of the electrical charges spread over the A and B cylindrical surfaces.

The Electrostatic Attraction Between any Pair of the A or B Cylinders is the Same.

Let us first suppose that the cylinders are external to one another. The attractive force on the A cylinder must remain constant if the electrostatic field surrounding it does not alter in magnitude or direction. The attraction on the A cylinder, therefore, is the same as if the B cylinder were a thin wire through B having a charge $-q$ per unit length. Since the attractions of the cylinders are equal and opposite, the required attraction will be equal to the attraction on this thin wire through B. But the attraction on this thin wire depends only on the number and direction of the lines of induction connected with it. It is independent, therefore, of the size of the A cylinder. Hence the attraction F per unit length between the two cylinders equals that between two thin wires having charges q and $-q$ per unit length respectively, and coincident with their centroid lines. It is therefore given by

$$F=\frac{2q^2}{\lambda\,.\,AB}=\frac{q^2}{\lambda r}, \quad \ldots \ldots \quad (19)$$

where λ is the inductivity of the dielectric.

If K be the capacity per unit length between the two cylinders and W the energy stored up in the dielectric, we have

$$W=q^2/(2\mathrm{K})=q^2\omega/\lambda.$$

Hence, $$F=\partial W/\partial c=(q^2/\lambda)(\partial\omega/\partial c).$$

Comparing this equation with (19), we see that

$$\frac{\partial\omega}{\partial c}=\frac{1}{r}. \quad \ldots \ldots \quad (20)$$

This equation can also be easily proved directly from the equations (4), (6) and (9), given above.

Similarly, when the sections of the cylinders are both A circles (Fig. 2), so that one of them is inside the other,

$$F=-\frac{q^2}{\lambda}\,\frac{\partial\omega_1}{\partial c}=\frac{q^2}{\lambda r}. \quad \ldots \ldots \quad (21)$$

The attraction, therefore, between any pair of cylinders which have the same inverse lines is the same.

The Stream Function.

Let u be the stream function corresponding to the potential function v. If u be zero at L (Fig. 3), we have

$$u=\int_0^\theta \sigma a \partial\theta$$

$$=\frac{q}{2\pi}\int_0^\theta \frac{\sinh a}{\cosh a-\cos\theta}\partial\theta$$

$$=\frac{q}{\pi}\tan^{-1}\frac{\tan(\theta/2)}{\tanh(a/2)}$$

$$=\frac{q}{2\pi}(\pi-\varphi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where θ is the angle PCL and φ is the angle APB.

It follows, as we have shown above, that every line of force such as PQ (Fig. 3) is part of a circle which passes through

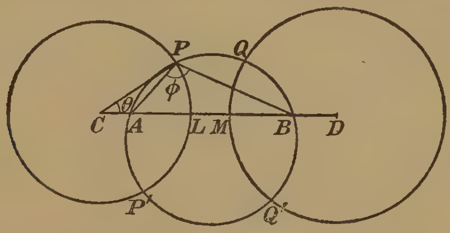


FIG. 3.
Every circle through A and B cuts the circles PLP' and QMQ' at right angles.

A and B. The capacity k_1 of the field per unit length between the portions PP' and QQ' of the cylinders intercepted by any one of these circles is given by

$$k_1=\lambda\,\frac{\frac{q}{2\pi}(\pi-\varphi)+\frac{q}{2\pi}\varphi}{v_1-v_2}=\lambda\frac{\frac{q}{2}}{2q\omega}=\frac{\lambda}{4\omega}. \quad . \quad . \quad . \quad (23)$$

The capacity k_1 is therefore equal to half of the capacity between the cylinders.

If R be the resistance per unit length between the two cylinders supposed of infinite conductivity, and if ρ be the resistivity of the medium between them, we have

$$KR=\rho\lambda/(4\pi),$$

where K is the corresponding capacity. Hence

$$R=\rho\omega/(2\pi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the resistance R_1 of the medium between PP' and QQ' (Fig. 3) would be given by

$$R_1 = \rho\omega/(4\pi)^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Similarly, if K' denote the thermal conductance between the two cylinders, and k' be the thermal conductivity, we have

$$K' = (4\pi/\lambda)k'K = 2\pi k'/\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and the thermal conductance between PP' and QQ' (Fig. 3 will be $K'/2$.

Cylinders with Unequal Charges.

Since the sum of the electric charges in a self-contained system must always be zero, it follows that if the sum of the charges on the cylinders be not zero there must be other charged conductors in the system. To fix our ideas we shall suppose that the axis of the cylinder whose radius is a, is also the axis of a very large hollow cylinder, the inner radius of which is c. If v_3 be the potential of this outer cylinder which surrounds the other two, we have

$$v_1 = p_{11}q_1 + p_{12}q_2 + p_{13}q_3,$$
$$v_2 = p_{21}q_1 + p_{22}q_2 + p_{23}q_3,$$
and
$$v_3 = p_{31}q_1 + p_{32}q_2 + p_{33}q_3,$$

where p_{11}, p_{12}, ... are the values of Maxwell's potential coefficients per unit length. If $v_1 = v_2 = v_3$ there can be no charges on the inner cylinders, and thus both q_1 and q_2 are zero. It follows, therefore, that $p_{13} = p_{23} = p_{33}$. Hence

$$v_3 = p_{33}(q_1 + q_2 + q_3).$$

We see that $1/p_{33}$ is the outside capacity per unit length of the hollow cylinder, and by taking this cylinder large enough it can be made as large as ever we please. Hence if we assume that the cylinders are surrounded by a co-axial hollow cylinder at a great distance from them, we can write $p_{13} = p_{23} = p_{33} = 0$. Our equations simplify to

$$v_1 = p_{11}q_1 + p_{12}q_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$
and
$$v_2 = p_{22}q_2 + p_{12}q_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

In the particular case when $q_1 = -q_2 = q$, we have by (7) and (8)

$$v_1 = 2q\alpha, \text{ and } v_2 = -2q\beta.$$

* Cf. G. Carey Foster and O. J. Lodge, "Proc. Phys. Soc.," Vol. I., . 115, 1875.

Hence, substituting in the simplified equations, we get

$$p_{11}=p_{12}+2\alpha, \quad \text{and} \quad p_{22}=p_{12}+2\beta \quad . \quad . (29)$$

Solving the equations for q_1 and q_2 in terms of v_1 and v_2 we get

$$q_1=k_{11}v_1+k_{12}v_2 \quad \text{and} \quad q_2=k_{22}v_2+k_{12}v_1,$$

where $k_{11}=p_{22}/\Delta$, $k_{22}=p_{11}/\Delta$, $k_{12}=-p_{12}/\Delta$, and $\Delta=p_{11}p_{22}-p_{12}^2$. The coefficients of v_1 and v_2 are called the capacity coefficients, and by considering the case when $q_1=-q_2$, we see that

$$k_{11}=\frac{1}{2\alpha}+\frac{\beta}{\alpha}k_{12}, \quad \text{and} \quad k_{22}=\frac{1}{2\beta}+\frac{\alpha}{\beta}k_{12}. \quad . \quad . \quad (30)$$

We also have

$$k_{11}=\frac{1}{2\alpha+2\beta p_{12}/(2\beta+p_{12})}, \quad \text{and} \quad k_{12}=\frac{-1}{2(\alpha+\beta)+4\alpha\beta/p_{12}}.$$

Since p_{12} is positive, we see that

k_{11} must lie in value between $1/(2\alpha)$ and $1/(2\alpha+2\beta)$,

k_{22} must lie in value between $1/(2\beta)$ and $1/(2\alpha+2\beta)$,

and that $1/2(\alpha+\beta)$ is a superior limit to $-k_{12}$. For instance, if a be very great compared with b, α will be very small compared with β, and thus k_{22} will equal $1/(2\beta)$ very approximately.

From the equations (27), (28) and (29) we deduce that

$$\begin{aligned} v_1-v_2&=2(q_1\alpha-q_2\beta) \\ &=(q_1+q_2)(\alpha-\beta)+(q_1-q_2)(\alpha+\beta). \quad . \quad . \quad (31) \end{aligned}$$

Whatever may be the values of the charges on the conductors this relation always holds. When $q_1+q_2=0$, it gives the capacity between the conductors, and when $q_1=q_2=q$ we have

$$\frac{q}{v_1-v_2}=\frac{1}{2(\alpha-\beta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Comparing this with equation (15) we see that when the charges on the cylinders are equal, the ratio of the charge to the difference of potentials is the same as for a cylinder of radius a and an enveloping cylinder of inner radius b, provided that the circular cross-sections have the same inverse points in the two cases. The energy stored in the field, however, in the latter case is $(v_1-v_2)^2/4(\alpha-\beta)$, whilst in the former case it is $(v_1^2-v_2^2)/4(\alpha-\beta)$.

When the charge on the B cylinder is zero

$$v_1-v_2=2q_1\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

In this case $q_1/(v_1-v_2)$ is a constant $(1/2a)$ which can be easily found. It is interesting to notice that the value of this constant is the same whichever of the B cylinders is chosen.

A Cylinder Inside a Hollow Cylinder, their Axes being Parallel.

If q_1, q_2, and v_1, v_2 be the charges and potentials of the cylinder (radius a) and sheath (inside radius b) respectively, it is easy to show that

$$q_1=\frac{1}{2(\alpha-\beta)}v_1-\frac{1}{2(\alpha-\beta)}v_2, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and

$$q_2=\left\{C+\frac{1}{2(\alpha-\beta)}\right\}v_2-\frac{1}{2(\alpha-\beta)}v_1, \quad . \quad . \quad (35)$$

where C is the capacity per unit length of the outer cylinder with respect to external bodies.

Hence also,

$$v_1=\left\{\frac{1}{C}+2(\alpha-\beta)\right\}q_1+\frac{1}{C}q_2, \quad . \quad . \quad . \quad . \quad (36)$$

and

$$v_2=\frac{1}{C}q_2+\frac{1}{C}q_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Hence when C can be found, we know the complete solution. We see that

$$k_{11}=-k_{12}=\frac{1}{2(\alpha-\beta)}=C_0 \text{ (say)}, \quad . \quad . \quad . \quad (38)$$

and

$$k_{22}=C+C_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Also

$$p_{11}=\frac{1}{C}+\frac{1}{C_0}, \qquad p_{22}=p_{12}=\frac{1}{C}. \quad . \quad . \quad . \quad (40)$$

If W denote the electrostatic energy,

$$\begin{aligned} W&=\tfrac{1}{2}p_{11}q_1^2+p_{12}q_1q_2+\tfrac{1}{2}p_{22}q_2^2 \\ &=\frac{(q_1+q_2)^2}{2C}+\frac{q_1^2}{2C_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41) \end{aligned}$$

Hence we deduce the following three theorems :—

(a) If q_1+q_2 is constant, W is a minimum when q_1 is zero, and, therefore, by (34) when $v_1=v_2$.

(b) If q_1=a constant, W is a minimum when $q_2=-q_1$, and in this case by (37), $v_2=0$.

(c) If q_2=a constant, W is a minimum when

$$q_1=-\{C_0/(C+C_0)\}\,q_2, \text{ and then by (36), } v_1=0.$$

We also have
$$W=\tfrac{1}{2}k_{11}v_1^2+k_{12}v_1v_2+\tfrac{1}{2}k_{22}v_2^2$$
$$=\tfrac{1}{2}C_0(v_1-v_2)^2+\tfrac{1}{2}Cv_2^2 \quad \ldots \ldots \quad (42)$$

Hence—

(a) If v_1-v_2 is constant, W is a minimum when $v_2=0$, and therefore when $q_2=-q_1$.

(b) If v_2 is constant, W is a minimum when $v_1=v_2$. In this case $q_1=0$.

(c) If v_1 is constant, W is a minimum when $v_2=\dfrac{C_0}{C+C_0}v_1$.

In this case $q_2=0$. A study of these theorems is instructive.

The force F per unit length between the cylinders when λ is the inductivity of the dielectric is given by

$$F=-\frac{q_1^2}{\lambda r}=-\frac{\lambda(v_1-v_2)^2}{4(\alpha-\beta)^2 r}. \quad \ldots \ldots \quad (43)$$

The equilibrium is unstable when the cylinders are co-axial, both when the charge on the inner cylinder or when the potential difference between them is maintained constant. In the former case they move so that the potential energy stored in the field is diminished and in the latter case so that it is increased.

Approximate Values of the Electrostatic Coefficients for Parallel Cylinders.

As a preliminary to finding approximate values for the potential coefficients in equations (27) and (28), let us consider the case of a concentric main, the radius of the inner cylinder being a, and the inner radius of the outer being d. The potential v at a point P between the cylinders distant r from the common axis is given by

$$v=2q_1 \log \frac{d}{r}, \quad \ldots \ldots \quad (44)$$

where q_1 is the charge per unit length on the inner cylinder. We see that for a given value of q_1 the greater the value of d, the greater will be the value of the coefficients of q_1 in this equation. If d is infinite, v is also infinite. This follows

because the work done in taking unit charge from the infinite cylinder to infinity is infinite.

Let us now consider the case of a very long charged prolate spheroid, the other conductor being a confocal spheroid (practically a sphere) at infinity. If l be the length of the axis of the spheroid and v be the potential at a point P on the equatorial plane at a distance r from the axis, we have

$$v=2q_1 \log \frac{l}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

very approximately, where q_1 is the charge on the surface intercepted between any two planes perpendicular to the axis and at unit distance apart.

Formulæ (44) and (45) prove that the actual value of the potential at a point near a charged cylinder even when it is at a great distance away from the ends of the cylinder depends both on the length of the cylinder and on the location of the necessary complementary charge. We notice, however, that the electric force at the point P is to a high degree of approximation independent both of the length and of the position of the complementary charge. Similarly if we have two parallel cylinders at a great distance away from the conductors carrying their complementary charge, we infer that the values of the potential coefficients will vary both with the length of the cylinders and with the position of the other conductors. We are led to infer also that the force per unit length between the cylinders is practically independent of their length and of the location of the complementary charge.

Considering now the case of the concentric main, let us suppose that d is very large and that we have a thin uncharged wire parallel to the axis of the main at a distance c from it, where c is great compared with a but very small compared with d. Since the field is practically undisturbed by the presence of this thin wire, its potential v_2 will be given by

$$v_2=2q_1 \log \frac{d}{c}.$$

Hence we see that

$$p_{12}=2 \log \frac{d}{c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

and therefore by (29)

$$p_{11}=2\alpha+2 \log \frac{d}{c}; \quad \text{and} \quad p_{22}=2\beta+2 \log \frac{d}{c}. \quad . \quad (47)$$

If we make d infinite, the potential coefficients become infinite, but the capacity coefficients are given by

$$k_{11}=k_{22}=-k_{12}=\frac{1}{2(\alpha+\beta)}=\frac{1}{2\log(c^2/ab)} \quad . \quad . \quad (48)$$

approximately.

The Electrostatic Forces Between the Cylinders.

From (27), (28), (46) and (47) we get

$$v_1=2\left(\alpha+\log\frac{d}{c}\right)q_1+2\log\frac{d}{c}\cdot q_2 \quad . \quad . \quad . \quad (49)$$

and

$$v_2=2\log\frac{d}{c}\cdot q_1+2\left(\beta+\log\frac{d}{c}\right)\cdot q_2 \quad . \quad . \quad . \quad (50)$$

Hence if λ be the inductivity, the electromagnetic energy W stored in the field is given by

$$W\lambda=\frac{1}{2}\left(2\alpha+2\log\frac{d}{c}\right)q_1{}^2+2\log\frac{d}{c}\cdot q_1q_2+\frac{1}{2}\left(2\beta+2\log\frac{d}{c}\right)q_2{}^2 \quad (51)$$

$$=\alpha q_1{}^2+\beta q_2{}^2+\log\frac{d}{c}\cdot(q_1+q_2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Hence

$$F=\frac{\partial W}{\partial c}$$

$$=\frac{1}{\lambda}\left\{q_1{}^2\frac{\sinh\alpha\cosh\beta}{r\sinh\omega}+q_2{}^2\frac{\cosh\alpha\sinh\beta}{r\sinh\omega}-\frac{(q_1+q_2)^2}{c}\right\} \quad . \quad (53)$$

$$=\frac{1}{2\lambda c^2r}\{q_1{}^2(c^2+b^2-a^2-2cr)+q_2{}^2(c^2+a^2-b^2-2cr)-4q_1q_2cr \quad (54)$$

By Kelvin's theorem (l.c. *ante*, p. 150) the conductor must move so as to diminish the electrostatic energy W. Hence when F is positive, that is when W increases with c the force is attractive, and when F is negative it is repulsive. We may write equation (54) in the form

$$F=\frac{1}{2\lambda c^2r}(Aq_1-Bq_2)(Aq_1-Cq_2), \quad . \quad . \quad . \quad (55)$$

where $A^2=c^2+b^2-a^2-2cr,$

$$B=(2cr+N)/A, \quad C=(2cr-N)/A,$$

and $N^2=(b^2-a^2)^2+c^3(4r-c).$

We see that when q_1 and q_2 are of opposite signs, F is always positive and, therefore, the force is always attractive. When, however, q_1 and q_2 have the same sign F is attractive, zero or repulsive according as q_1/q_2 is greater than B/A or less than C/A, equals either of these quantities or lies in value between them.

Particular cases are of interest. If $q_1=-q_2=q$ (54) reduces to (19), which is exactly true. If, in addition $a=b$, we have

$$F=\frac{2q^2}{\lambda(c^2-4a^2)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

When $q_1=q_2=q$, the formula reduces to

$$F=-\frac{(4r-c)q^2}{\lambda cr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

In the particular case when the B cylinder is so thin that we may write $b=0$, and therefore $2cr=c^2-a^2$, we get

$$F=\frac{2q_2\{a^2q_2-(c^2-a^2)q_1\}}{c(c^2-a^2)}. \quad . \quad . \quad . \quad . \quad (58)$$

If q_2 is zero, F vanishes. When q_1 is zero the attractive force is given by

$$F=\frac{2a^2q^2{}_2}{c(c^2-a^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

We see from (58) that when q_1 and q_2 have the same sign the force is attractive, when q_2/q_1 is greater than $(c^2-a^2)/a^2$;

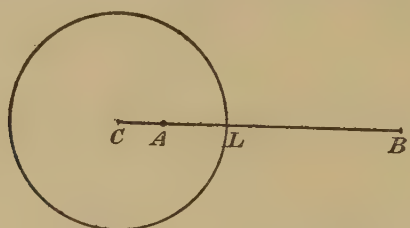


FIG. 4.

$CA.\ CB=CL^2.$

The wire which is the image of the wire through B passes through A.

it vanishes when q_2/q_1 has this value and it is repulsive, when q_2/q_1 is less than $(c^2-a^2)/a^2$.

A direct geometrical proof of (58) can easily be given by the method of images as follows :—

Let C and B (Fig. 4) be the centres of the cross-sections of the cylinder and wire respectively. If the charge per unit

length on the wire be q_2, its image in the cylinder will be a wire through A having a charge $-q_2$ per unit length, where $CA \,.\, CB=a^2$. Hence if the total charge on the cylinder be q_1 per unit length we can replace the cylinder by two parallel wires through C and A respectively, which have charges q_1+q_2 and $-q_2$ per unit length respectively. Hence we see at once that the force F is given by

$$F=\frac{2q_2(q_1+q_2)}{c}-\frac{2q_2^2}{c-a^2/c}$$

$$=\frac{2q_2\{a^2q_2-(c^2-a^2)q_1\}}{c(c^2-a^2)},$$

which is equation (58).

High Frequency Currents.

We know that at very high frequencies the currents distribute themselves over the surface of the cylinders in such a way that there are no magnetic lines of force produced in the metal. We thus see that when the currents are equal to $+I$ and $-I$ the current density i per unit of the circumference of the A cylinder is given by

$$i=\frac{I}{2\pi a}\cdot\frac{\sinh\alpha}{\cosh\alpha-\cos\theta}. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

See Fig. 2 and compare with equation (11).

Similarly for the B cylinder,

$$i=-\frac{I}{2\pi b}\cdot\frac{\sinh\beta}{\cosh\beta-\cos\theta}. \quad . \quad . \quad . \quad . \quad (61)$$

Since $l/r=1-e\cos\theta$ is the equation to an ellipse referred to the focus as origin and the major axis as initial line, l being the semi latus rectum and e the eccentricity, we see that the length Cp of the radius vector of the ellipse MpM' in Fig. 5 gives the current density i at the point P on the cylinder A when carrying high frequency currents, the eccentricity of the ellipse being sech α and its major axis $(I/\pi a)\coth\alpha$.

The ratio of the greatest to the least current density on the A cylinder equals CM/CM', which equals $\coth^2(\alpha/2)$.

Let us now consider the case when a potential difference e_1 per unit length is applied to the A cylinder and a P.D. e_2 per unit length to the B cylinder. If I_1 and I_2 are the currents

produced on the cylinders, then, since with high frequencies the resistance terms can be neglected, our equations are

$$e_1=L_{11}\frac{\partial I_1}{\partial t}+L_{12}\frac{\partial I_2}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad (62)$$

and

$$e_2=L_{22}\frac{\partial I_2}{\partial t}+L_{12}\frac{\partial I_1}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad (63)$$

where L_{11}, L_{22} and L_{12} are the inductance coefficients with high frequency currents.

Integrating these equations we get

$$\Phi_1=L_{11}I_1+L_{12}I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and

$$\Phi_2=L_{22}I_2+L_{12}I_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

where Φ_1 and Φ_2 are the linkages of the lines of induction with the currents in the A and B cylinders respectively.

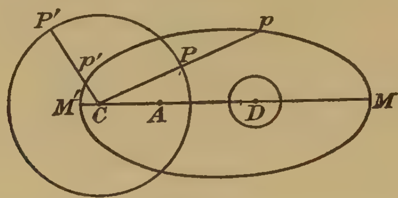


Fig. 5.

The current density at the point P on the cylinder equals Cp, where C is the focus of the ellipse whose major axis is in the same line as $C\,D$, and for which $C\,M=(I/2\pi a)\coth(\alpha/2)$; $C\,M'=(I/2\pi a)\tanh(\alpha/2)$ and the eccentricity$=\operatorname{sech}\alpha=\dfrac{2\,c\,a}{c^2+a^2-b^2}$.

Let us first suppose that the cylinder A is inside the cylinder B. Since the currents are so distributed that the resultant magnetic force inside the metal of either conductor is zero we see by comparing (36) and (37) with (64) and (65) that

$$L_{11}=1/C+2(\alpha-\beta), \quad \text{and} \quad L_{12}=1/C=L_{22}. \quad (66)$$

If $I_1=-I_2$ the self inductance L, per unit length, of the circuit formed by the two cylinders is given by

$$L=L_{11}+L_{22}-2L_{12}=2(\alpha-\beta). \quad . \quad . \quad . \quad . \quad (67)$$

When the circuits are in parallel $e_1=e_2$, and hence

$$I_1=\frac{L_{22}-L_{12}}{L_{11}+L_{22}-2L_{12}}(I_1+I_2)=0 \quad . \quad . \quad . \quad . \quad (68)$$

since $L_{22}-L_{12}=0$. Thus all the current flows on the outside of the outer conductor.

If the outer cylinder is insulated so that $e_2=0$, we have $L_{22}I_2+L_{12}I_1=0$, and so $I_1=-I_2$. The induced current on the outer cylinder is thus equal and opposite to the current on the inner cylinder. Comparing also with the analogous electrostatic problem, we see that it is on the inside of the outer cylinder.

If the inner cylinder is insulated so that $e_1=0$, we have

$$I_1=-\frac{L_{12}}{L_{11}}I_2=-\frac{C_0}{C+C_0}I_2.$$

On the inner surface of the outer cylinder, therefore, there must flow a current $C_0I_2/(C+C_0)$, and on the outer surface a current $CI_2/(C+C_0)$. When this occurs the electromagnetic energy

$$\tfrac{1}{2}L_{11}I_1^2+L_{12}I_1I_2+\tfrac{1}{2}L_{22}I_2^2 \quad . \quad . \quad . \quad . \quad (69)$$

has a minimum value, since I_2 is constant.

Since L_{12} and L_{22} are independent of c, the distance between the axes of the cylinders, the force F per unit length between them is given by

$$F=\tfrac{1}{2}\frac{\partial L_{11}}{\partial C}I_1^2=\tfrac{1}{2}\frac{\partial}{\partial C}\left\{\frac{1}{C}+2(\alpha-\beta)\right\}I_1^2=-\frac{I_1^2}{r} \quad . \quad . \quad (70)$$

the equilibrium being stable when the cylinders are co-axial.

When the cylinders are solid and parallel to one another we have

$$e_1=\left(2\alpha+2\log\frac{d}{c}\right)\frac{\partial I_1}{\partial t}+2\log\frac{d}{c}\cdot\frac{\partial I_2}{\partial t} \quad . \quad . \quad . \quad (71)$$

and

$$e_2=\left(2\beta+2\log\frac{d}{c}\right)\frac{\partial I_2}{\partial t}+2\log\frac{d}{c}\cdot\frac{\partial I_1}{\partial t} \quad . \quad . \quad . \quad (72)$$

approximately. The greater the values of d/c, c/a and c/b the more accurate will be the equations. Hence

$$L_{11}=2\alpha+2\log\frac{d}{c};\quad L_{22}=2\beta+2\log\frac{d}{c};\quad L_{12}=2\log\frac{d}{c}.$$

Thus the self inductance L per unit length of the circuit formed by the two cylinders in series will be given by

$$L=L_{11}+L_{22}-2L_{12}=2(\alpha+\beta) \quad . \quad . \quad . \quad . \quad (73),$$

an equation which is exactly true at all distances apart.

Comparing (10) and (73) we see that $LK=1$.

If W be the electromagnetic energy stored in the field at any instant, we have

$$W=\alpha I_1^2+\beta I_2^2+\log\frac{d}{c}\cdot(I_1+I_2)^2 \quad . \quad . \quad . \quad (74)$$

When the cylinders are in parallel we get

$$I_1=\frac{\beta}{\alpha+\beta}(I_1+I_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

or

$$I_2=\frac{\alpha}{\alpha+\beta}(I_1+I_2).$$

At every instant, therefore, the ratio I_1/I_2 equals β/α. Since $\beta/\alpha=\sinh^{-1}(r/b)/\sinh^{-1}(r/a)$, we see that if b is greater than a, I_1/I_2 is less than unity. Hence the smaller conductor carries the smaller current. The density of the current on the smaller conductor is also less than on the larger conductor. It is to be noticed that (75) is the condition that (74) has a minimum value when I_1+I_2 is a constant.

If F be the instantaneous value of the E.M.F. acting on the cylinders per unit length, we have

$$F=\frac{\partial W}{\partial c}=I_1^2\frac{\sinh\alpha\cosh\beta}{r\sinh\omega}+I_2^2\frac{\cosh\alpha\sinh\beta}{r\sinh\omega}-\frac{(I_1+I_2)^2}{c} \qquad (76)$$

$$=\frac{1}{2c^2r}\{I_1^2(c^2+b^2-a^2-2cr)+I_2^2(c^2+a^2-b^2-2cr)-4crI_1I_2\} \qquad (77)$$

$$=\frac{1}{2c^2r}(AI_1-BI_2)(AI_1-CI_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

where A, B and C have the same values as in (55).

Now by Kelvin's theorem* if I_1 and I_2 are maintained constant the conductors move so as to increase the potential energy. Since W increases with c when F is positive, it follows that when F is positive the force is repulsive, and when negative it is attractive. We deduce from (78) that when I_1 and I_2 are of opposite signs the force is always repulsive. When, however, the currents I_1 and I_2 are flowing in the same direction the force is attractive when I_1/I_2 lies in value between B/A and C/A. When the ratio of these currents equals either of these limiting values the force vanishes, and when it lies outside those limits the force is repulsive.

In practice we are concerned with the average value of the force F taken over a whole period. If φ be the phase difference between the currents I_1 and I_2, if F' denote the average value of F, and I_1' and I_2' be the effective values of the currents, we have by (78)

$$F'=\frac{1}{2c^2r}\{A^2I_1'^2+B^2I_2'^2-A(B+C)I_1'I_2'\cos\varphi\}. \quad . \quad (79)$$

* Russell's "Alternating Currents," Vol. I., p. 41.

When $\cos\phi=1$ or -1, the equation is practically identical with (78). In any case, however, it can easily be put into factors and discussed in a similar way.

In the particular case when $I_1=I_2=I$, we have $\cos\phi=1$, and

$$F'=-\frac{I^2}{cr}(4r-c), \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

the force being always attractive.

When $I_1=-I_2=I$, $\cos\phi=-1$, and hence

$$F'=\frac{I^2}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

the force being repulsive. It is to be noticed that (80) is only an approximate formula, while (81) is an exact formula.

When the cylinder B is a very fine wire, formulæ (76) to (78) are extremely accurate. Putting $b=0$ and $2cr=c^2-a^2$, in 77 we get

$$F=\frac{2I_2\{a^2I_2-(c^2-a^2)I_1\}}{C(c^2-a^2)}. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

It can be readily shown by the method of images that this formula is exact. We have also

$$F'=\frac{2I_2'\{a^2I_2'-(c^2-a^2)I_1'\cos\phi\}}{C(c^2-a^2)}. \quad . \quad . \quad (83)$$

We see that when I_2'/I_1' is greater than $(c^2-a^2)\cos\phi/a^2$ the force is repulsive. When this ratio equals $(c^2-a^2)\cos\phi/a^2$ the force vanishes, and when it is less than this value it is attractive.

It follows also from (83) that when the distance between the wire and the axis of the cylinder is less than

$$a\{(I_1'\cos\phi+I_2')/I_1'\cos\phi\}^{\frac{1}{2}}$$

the force is repulsive, when it is equal to it the force vanishes, and when it is greater than this value it is attractive.

Hence when

$$c=a\{(I_1'\cos\phi+I_2')/I_1'\cos\phi\}^{\frac{1}{2}} \quad . \quad . \quad . \quad (84)$$

the wire is in a position of stable equilibrium.

In conclusion, physicists should bear in mind that the potential and capacity coefficients of conductors have perfectly determinate values. Even in simple cases values are difficult to find by calculation, but in every case they can be

found accurately by experiment. If we compare (27) and (28) with (64) and (65) we see at once that $L_{11}=p_{11}$, $L_{12}=p_{12}$ and $L_{22}=p_{22}$.* It follows that the inductance coefficients of conductors for high frequency alternating currents can be found very simply by determining experimentally the potential coefficients of the conductors for electrostatic charges.

ABSTRACT.

Many problems in connection with parallel cylindrical conductors occur in practical electrical work. The formulæ for the capacity between the conductors and for the effective inductance are well known, but the values of the capacity and potential coefficients and of the inductance coefficients have not yet been determined. It is shown that for the case of a cylinder inside a cylindrical tube their values can in all cases be easily computed. When the cylinders are external to one another, it is proved that the three capacity coefficients are connected by two very simple relations. Limiting values between which these coefficients must lie are found, and methods of obtaining closely approximate values in special cases are given. Whatever the charges on the cylinders may be, provided that the other conductors of the system are remote, the mutual force between them can be calculated to high accuracy when their distance apart is great or when the radius of one is small compared with that of the other.

Practically identical formulæ enable us to find the current-density and the inductance coefficients with high-frequency currents, both for a cylinder inside a cylindrical tube and for two parallel cylinders. In the latter case it is shown that when the phase difference between the currents is less than 90 deg. the mechanical force between the cylinders is repulsive when they are close together and attractive when they are far apart. At a definite distance apart, therefore, the cylinders when carrying high-frequency currents are in stable equilibrium. Since the potential coefficients can always be determined experimentally, it follows that the inductance coefficients for high-frequency currents which are equal to them are also found by the same experiments.

DISCUSSION.

Dr. D. Owen said that in the first paragraph of the section on High Frequency Currents, the author said "We thus see—&c." He did not quite know the justification for this.

Mr. F. J. W. Whipple said it surprised him that the solutions to these problems had not all been already worked out. It appeared clear to him that one could write down the solutions in θ functions without much trouble. By so doing many difficulties might be got over since these functions were tabulated. Dr. Russell handled infinities rather familiarly, and he was not certain of the reliability of some of the solutions. Then he obtained a result for a thin wire and assumed it to hold for a thick one. One did not know how far the assumption was justified.

Dr. Russell, in reply, said the object of the Paper was to obtain approximate solutions to clear the ground for a complete discussion of the problem. Dr. Owen's point was explained in one of Kelvin's works, to which he would send him the reference.

* Cf. *l.c. ante*, Vol. I., p. 201.

XII. *Temperature Coefficient of Tensile Strength of Water.* *By* S. SKINNER, *M.A.*, *and* R. W. BURFITT, *B.Sc.*

RECEIVED DECEMBER 9, 1918.

IN a Paper on the effect of temperature on the hissing of water when flowing through a constricted tube (Roy. Soc., "Proceeding A," Vol. XCI., 1915, p. 481), Skinner and Entwistle described the measurements which indicated that the tensile strength of water became zero at a temperature of about 320°C., a temperature approaching the critical point of water. The conclusion was drawn that the tensile strength of water diminished with temperature to the critical point. To make the law general, it is necessary to examine the behaviour of other liquids, but the original form of the apparatus was quite inappropriate, since the water at a high pressure was drawn directly from the water supply of the building. It was with a view of designing a small apparatus in which other liquids could be tried that the following experiments were commenced.

Meanwhile Sir Joseph Larmor ("Proc." Lond. Maths. Soc., 1916, p. 191), on the assumption that the van der Waals form of equation holds in the liquid state, arrived at the theoretical conclusion that the negative pressure could subsist only at absolute temperatures below 27/32 of the critical point of the substance. For water the critical point is 365°C.; thus in this substance, internal tension could persist to 538°Abs., or 265°C. This conclusion was of interest, since the experiments quoted above had shown that the tensile strength probably disappeared at about 320°C., and from what follows, this result appears to be more nearly 245°C., which is in agreement with the theory.

In the new experiments the tube which contained the water consisted of two vertical wide-bore arms, length 22 cm., diameter 3 cm., connected by a tube, length 12 cm., the central portion of which was a capillary of length 2·5 cm., and bore 1·195 mm. diameter. The arrangement of the apparatus is shown in the diagram. We shall now describe the method of making the experiments.

Air was pumped into a cylinder *R*, of some 6 litres capacity, which was kept at a constant temperature by a water jacket,

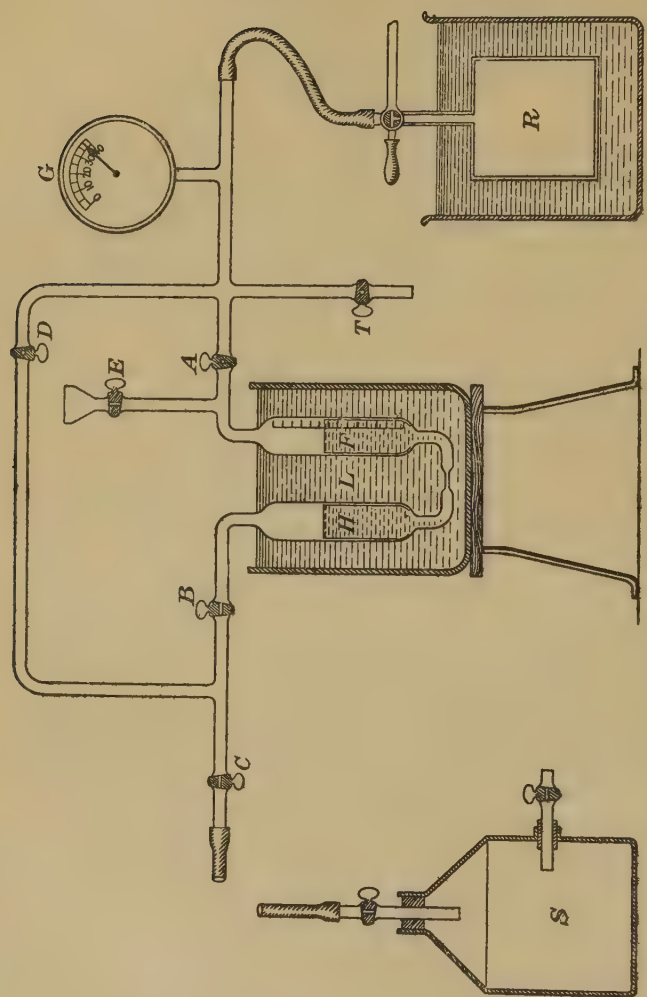


FIG. 1.—DIAGRAM OF APPARATUS.

Fig. 1. From this reservoir, the air pressure could be exerted along different paths by the adjustment of taps. A gauge, *G*, recorded the pressure of the air. The taps *A*, *B* being turned on, and the tap *C* being open to the air, and *D*, *E* and *T* being shut, the air forced the liquid from limb *F* of experimental tube through constriction into limb *H*. When taps *C*, *A* and *T* were shut and *E* open, the air could be used to drive the liquid back to its initial position. By trial it was possible to make a nice adjustment of the pressure of the air in *R*, so that the speed for rupture could be instantly given to the liquid. The rupture of the liquid was generally judged by the sound, although the appearance of the liquid as regards cloudiness produced by little drops was also noted. Along the arm *F* a vertical scale 15 cm. long was fixed. A stop-watch reading to one-fifth of a second was used to obtain the time required for the liquid to descend through a particular range, and thus the relative speeds in the constriction could be deduced.

The air reservoir *S*, with its two taps, could be attached when a back pressure was required. This may be necessary when the temperature of the liquid under examination is near the boiling point. Moreover, the use of a back pressure enables observations to be made at temperatures beyond the normal boiling point.

It will be observed that in this form of apparatus the same sample of liquid is used over and over again. In this, the experiments differ from the former, in which fresh liquid was used for each observation. Moreover, the quantity of liquid required is only small, which is necessary if the method is to be used for liquids other than water.

A large number of observations, with the tap *C* being open to the air, were made up to a temperature near 90°C., and between 90° and 100°C. with the aid of a back pressure in the reservoir *S*. These observations are shown in a diagram, Fig. 2, and a straight line drawn through them cuts the axis at a temperature near 245°C.

The actual observations for velocity and temperature are recorded in the table and also columns are given, one containing the pressure indicated by the gauge, and another giving the value of the coefficient of velocity when the velocity at 250°C. is assumed to be zero.

This result confirms the general conclusion obtained in the paper of Skinner and Entwistle, and is in agreement with

Sir Joseph Larmor's theoretical views. If V is the velocity of the current at the constriction at a temperature t°C., and θ°C. the critical temperature of water and C a constant, then

$$V=C\left\{\frac{27}{32}(\theta+273)-(t+273)\right\}$$

The experiments show that this formula is true for temperatures between 0°C. and 100°C.

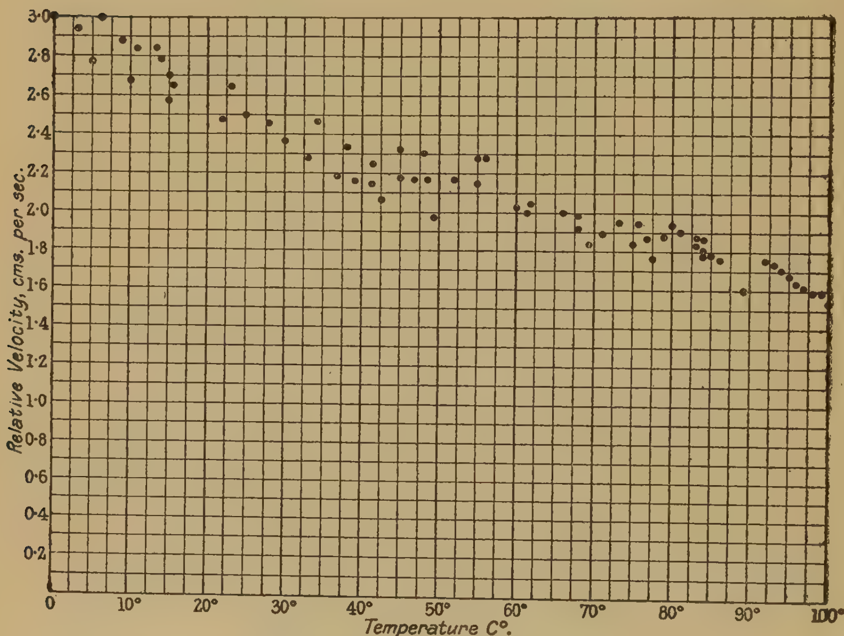


FIG. 2.—GRAPH OF TEMPERATURE VELOCITY, OCTOBER, 1917–18.

We are now making a much stronger apparatus for the study of oils, and some preliminary experiments appear to indicate that a similar law holds for them.

TABLE OF RESULTS.

Date.	*t.* Temperature.	*a.* Proportional Velocity.	*b.* Pressure in lb. per sq. in.	$\frac{a}{250-t}$
	Deg. C.	Cms.	Lbs.	
Oct. 29, 1917	15·5	2·65	29·5	0·01133
	22·0	2·48	24·5	0·01087
	33·0	2·28	20·0	0·01051
	37·0	2·16	19·0	0·01028
	39·0	2·19	18·0	0·01023
Nov. 5, 1917	30·0	2·37	21·5	0·01076
	38·0	2·33	19·5	0·01099
	41·5	2·24	18·5	0·01072
	45·0	2·18	17·5	0·01064
	48·5	2·17	17·5	0·01076
	52·0	2·16	17·0	0·01091
Nov. 10, 1917	55·0	2·15	16·0	0·01103
	61·5	2·00	15·5	0·01060
	69·5	1·83	15·0	0·01014
	75·5	1·94	14·8	0·01112
	81·0	1·89	14·4	0·01119
	83·0	1·86	14·2	0·01114
Nov. 12, 1917	66·0	2·03	16·4	0·01069
	68·0	1·91	15·2	0·01049
	77·0	1·87	14·5	0·01081
	83·0	1·82	13·5	0·01090
Nov. 15, 1917	73·0	1·94	15·1	0·01096
	79·0	1·88	14·0	0·01100
	85·0	1·78	12·5	0·01078
Nov. 28, 1917	75·0	1·83	15·0	0·01046
	84·0	1·80	13·0	0·01100
	86·0	1·75	12·5	0·01067
	62·0	2·04	16·8	0·01085
	47·0	2·17	18·5	0 01069
Nov. 29, 1917	66·0	2·00	15·5	0·01086
	80·0	1·92	14·8	0·01129
	68·0	1·98	15·2	0·01087
	56·0	2·28	16·4	0·01175
	48·0	2·30	17·4	0·01139
	45·0	2·31	18·2	0·01127
Dec. 14, 1917	13·5	2·84	36·0	0·01200
	34·5	2·46	20·0	0·01090
	41·5	2·14	17·5	0·01026
	42·5	2·06	17·5	0·00995
	49·5	1·97	17·0	0·00984
	55·0	2·28	15·9	0·01169
	71·0	1·89	16·0	0·01055
	84·0	1·85	13·0	0·01115
	89·0	1·60	12·2	0·00996
Jan. 11, 1918	0·0	3·00	42·0	0·01200
	5·0	2·77	38·0	0·01131
	10·0	2·68	36·0	0·01116
	15·0	2·58	33·0	0·01097
	23·0	2·64	29·0	0·01163
	25·0	2·50	28·0	0·01106
	28·0	2·45	26·0	0·01104

TABLE OF RESULTS.—*Continued.*

Date.	*t.* Temperature.	*a.* Proportional Velocity.	*b.* Pressure in lb. per sq. in.	$\frac{a}{250-t}$
Jan. 26, 1918	0·0	3·00	42·0	0·01200
	3·0	2·94	38·0	0·01191
	9·0	2·88	36·5	0·01195
	11·0	2·83	36·0	0·01184
	14·0	2·78	34·5	0·01178
	15·0	2·70	33·0	0·01149
Feb. 23, 1918	92·0	1·75	15·0	0·01108
	93·0	1·72	15·0	0·01095
Mar. 9, 1918	94·0	1·70	14·0	0·01090
	95·0	1·66	13·0	0·01071
	96·0	1·63	12·0	0·01064
	97·0	1·61	11·0	0·01052
	98·0	1·59	8·0–9·0	0·01047
	99·0	1·59	8·0–9·0	0·01053
	100·0	1·53	8·0–9·0	0·01020

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ABSTRACT.

The liquid is forced under pressure through a capillary constriction between two limbs of a U-tube. By trial the pressure is adjusted until the speed in the capillary is sufficient to produce rupture. This is judged by the sound and also the appearance. The whole U-tube is immersed in a bath, the temperature of which can be varied. Actual observations of rupture, velocity and temperature are recorded up to about 100°C., from which it is deduced that the tensile strength becomes zero in the neighbourhood of 245°C., which is in agreement with theory.

DISCUSSION.

Dr. VINCENT asked what arrangements were made to get the liquid back into the virgin state after it had been churned up by forcing through the capillary.

Dr. BRYAN asked how dissolved gas affected the results. In experiments by Worthington and others the water required to be very carefully boiled before it showed any tensile strength at all.

Prof. LEES also commented on the fact that the liquid experimented on here contained dissolved gas. In Worthington's experiments with gas-free water the tensile stress was about 2 atmospheres, as far as he remembered. This seemed of a different order from that obtained by the authors.

Mr. F. J. W. WHIPPLE asked where the tensile strength came into the formula. The diagram seemed to connect velocity and temperature only.

Mr. BURFITT, in reply to Dr. Vincent, said the liquid was always allowed to stand until all cloudiness had disappeared, and any dissolved gas was in equilibrium. There was no doubt the presence of dissolved gas promoted rupture. Worthington's method was static, and there was no danger of air getting in if the liquid was initially air free. In the present method it was impossible to prevent some air becoming dissolved. With reference to Mr. Whipple's comments, the tables are not primarily intended to give actual values of the tensile stress, but to give its variation with temperature. If desired, the value of C in the theoretical formula can be calculated from the figures given.

XIII. *Vector Diagrams of Some Oscillatory Circuits Used with Thermionic Tubes.* By W. H. Eccles, *D.Sc.*

Received January 12, 1919.

The circuits used in generating electrical oscillations by aid of three-electrode thermionic tubes or relays are now well known, and need not be fully described here; but the general principles of operation may be briefly summarised.

The practical problem is how to sustain the electrical oscillations in a given oscillatory circuit by aid of a rapidly acting relay such as an ionic tube. Evidently from analogy with the balance wheel of a watch and its escapement this can be done by applying properly timed E.M.F.'s. Any relay can do this if the energy taken from the oscillator to operate the relay is small compared with the amount given to the oscillator by the action of the relay. To be more precise, the energy given to the oscillator must be greater than that taken to operate the relay by the amount expended in the oscillator as useful work and as waste.

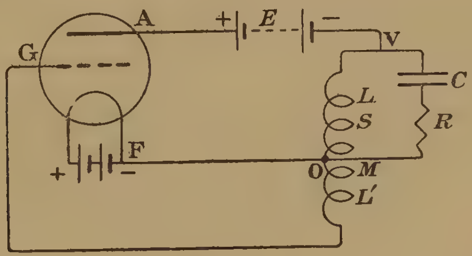


Fig. 1.

One of the simplest ways of associating an oscillatory circuit with an ionic tube is given in Fig. 1. Here the oscillator is made up of an inductance coil L and a condenser C connected in parallel. The common terminals of coil and condenser are connected to the filament and the anode of the tube, a battery of high voltage, E, being inserted in either lead to the tube. The battery causes electrons to flow from the filament to the plate, and if nothing else is done a steady current appears to flow into the tube at its anode and out at the filament and through the inductance coil L. This flow is necessary in order to set up the space charge upon which the grid of the tube acts when the tube is used as a relay.

In the mode of connection shown in the Figure the grid is acted upon by E.M.F. induced in the grid coil L' by means of its mutual inductance M with the oscillator coil. When the grid is made positive relative to the filament the current in the plate circuit increases, and when it is made negative the current decreases. The circuit FL'GF is called the control circuit, and the circuit FOVEAF is sometimes called the repeat circuit.

If we suppose an oscillation started in the circuit L, C an oscillatory E.M.F. is induced in L' and applied to G. Let its value be e_g; then by the properties of the tube this is transferred to the repeat circuit as an E.M.F. ge_g, where g may be called the voltage ratio of the tube and is of order about 10 in many tubes. If the mutual inductance is of the right sign this E.M.F. acts at the oscillator terminals OV in the right sense to assist the oscillatory current running at the instant, but if it is of the wrong sign the control E.M.F. tends to stop the oscillatory current. The sign of the mutual inductance is altered by reversing one of the coils. When the mutual inductance is of the right sign and is great enough, and if certain other conditions to be discussed are satisfied, the oscillation in L, C will be maintained, the energy expended in resistances R and S being supplied by the battery E.

I. Langmuir* gave the empirical equation

$$I=A(V_a+gV_g)^{3/2},$$

to represent the total current I through the tube from plate to filament when the voltage between plate and filament is V_a and that between grid and filament V_g. M. Latour† and G. Vallauri‡ introduced approximate equations for discussing small changes of voltage and current, and these may be written in the form

$$I_0+i=h_0+h_aV_a+h_gV_g,$$

where I_0 is the steady part of the current through the tube, i the variable part of the current, V_a, V_g the voltages (relative to the filament) applied to anode and grid, and h_0, h_a, h_g are constants of the tube having fairly definite values at a fixed temperature of filament. Usually V_a includes a constant and a variable portion, say E and e_a, and a similar statement holds

* I. Langmuir, "Gen. El. Rev.," XVIII., pp. 327-339, May, 1915.
† M. Latour, "Electrician," LXXVIII., pp. 280-282, December, 1916.
‡ G. Vallauri, "L Elettrotecnica," January 25–February 5, 1917.

for V_g, and then if we ignore the unvarying parts of the current and the voltage we obtain the equation

$$i=h_ae_a+h_ge_g,$$
$$=h_a(e_a+ge_g).$$

The quantities h_a and h_g are of the nature of conductances.

In the mode of connecting oscillator to tube shown in Fig. 1 the anode current when it is steady all passes through the inductance coil of the oscillator ; but when the anode current has a variable part this divides between the inductances and

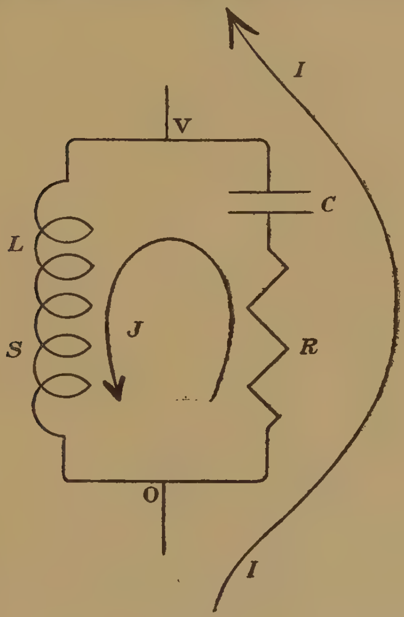


FIG. 2

the condenser branches. This division may be represented by the cyclic notation as indicated in Fig. 2. We shall at once assume the variable current to be of sine form, and shall therefore write

$$i=I \sin \omega t,$$
$$j=J \sin (\omega t+\theta).$$

On this assumption we shall draw the vector or crank diagram of the circuit. In Fig. 3 OU is taken in any direction on the paper to represent to scale the fall of potential along the

resistance R due to the current I alone, that is $OU = RI$. In accordance with the usual convention the changing values of the P.D. at the ends of the resistance R are given by rotating the vector OU counter-clockwise and taking its projection upon a fixed line in the plane of rotation. In the same way UD is drawn to represent the fall of potential through the condenser due to the current I; it is well known that this will be of magnitude $I/C\omega$ and 90 deg. behind the vector OU. But the circulating current J also traverses R and C; let the consequent potential fall in C be represented by $DQ = J/C\omega$, and that in R by $QV = RJ$. The current J flows also in the inductance L and resistance S. The fall of potential in S must be in phase with that due to the same current in R, and

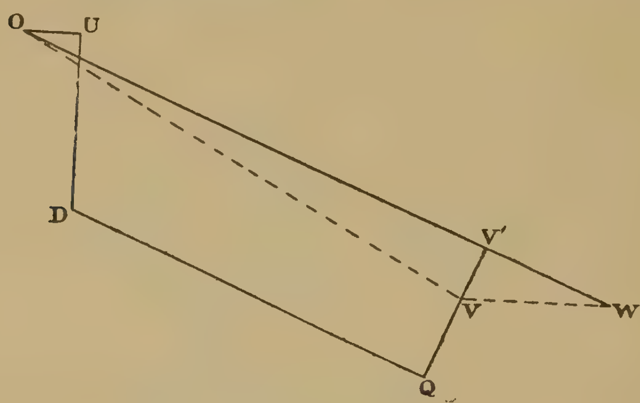


FIG. 3.

that in L must be in opposite phase to that in C and therefore VV′, the prolongation of QV is made equal to SJ and $V'O = L\omega J$. This last vector is made to end at O in order to fulfil the condition that in going round the closed circuit C, R, L, S the total fall of potential, reckoned with due regard to the phase in each element, must be zero since there is no source of E.M.F. in the closed circuit.

The points O and V on the crank diagram of Fig. 3 correspond to the points O and V in Fig. 2, and in fact the line joining O to V in the crank diagram represents the magnitude and phase of the fall of potential across the oscillator due to the sine current of amplitude I traversing it. Since the applied E.M.F. in the anode circuit of the tube is an unvarying one E,

the voltage actually applied to the plate is alternately less and greater than E to an extent determined by the position of the representative vector OV. Moreover, when the alternating current i flows through the tube the fall of potential between the plate and the filament inside the tube is, we have seen, equal to I/h_a in amplitude and, of course, in phase with the vector RI or OU. This new vector is drawn in Fig. 3 at VW, so as to add geometrically to OV. The result is OW. It is placed in the same direction as OV′ for a reason now to be explained.

It is plain that in order to maintain steady the oscillatory currents represented by the crank diagram the ionic relay must supply the alternating E.M.F. required to drive the current i through the oscillator and the tube, that is, it must supply the E.M.F. represented by OW. Hence the voltage applied to the grid must be an alternating voltage in phase with OW and of $1/g$th the magnitude. But in the circuit of Fig. 1 the grid voltage is supplied by induction from the coil L. We shall assume that the current flowing to the grid under this induced voltage is negligible—as is almost always permissible—and then the reaction of the induced current upon the primary current is also negligible. Then the voltage induced in the grid circuit is of amplitude $M\omega J$ and is in phase with the potential fall $L\omega J$ in the coil L, or in exact opposition, according to the sign of M. In Fig. 1 the mutual inductance is such that when the fall of potential is from O to V the induced voltage makes the grid positive with respect to the filament. We therefore measure along OV′ a length equal to M/L times OV′ in order to represent the E.M.F. applied to the grid. As already seen, the action of the tube is equivalent to the application in the plate circuit of a voltage g times that applied to the grid. But the E.M.F. required in the plate circuit is measured by OW in the crank diagram. Hence we must have

$$\mathrm{OW}=g\frac{M}{L}\mathrm{OV}'.$$

This condition and the condition that VW shall be parallel to OU, already discussed, are together sufficient and necessary for the maintenance of oscillations. In drawing the crank diagram these conditions have to be borne in mind.

From the diagram the formulæ relative to this mode of generating oscillations can be obtained. For this purpose the diagram is repeated in Fig. 4 with the lengths of the lines

marked for convenience. The additional line DN is perpendicular to OW and makes on OW an intercept,

$$\begin{aligned} ON &= OV'-DQ \\ &= (L\omega - 1/C\omega)J \\ &= XJ. \end{aligned}$$

The parallelism of OU and VW is indicated by marking the angles UOW and OWV each equal to φ.

Project the broken line OUD upon ON and ND. We obtain the two equations

$$XJ = RI\cos\varphi + (I/C\omega)\sin\varphi$$
$$(R+S)J = (I/C\omega)\cos\varphi - RI\sin\varphi.$$

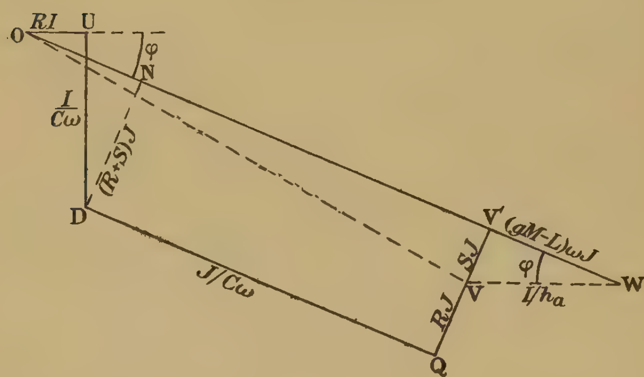


FIG. 4.

But from the triangle VV′W we have

$$\sin\varphi = \frac{SJ}{I/h_a}$$
$$\cos\varphi = \frac{(gM-L)\omega J}{I/h_a}.$$

Hence
$$X = Rh_a\omega(gM-L) + h_a S/C\omega,$$
$$R+S = h_a(gM-L)/C - h_a RS.$$

From these we obtain two other equations by solving for $gM-L$ and for ω in turn. The former solution yields

$$gM = L + C\ RS + \frac{R+S}{h_a}\Big),$$

which gives the magnitude of the factors concerned in the control—namely, g and M—in terms of the constants of the

oscillator and the conductance of the tube. If the product gM be smaller than required by this equation the oscillations will die away.

Now, solving the above pair of equations for ω, we obtain

$$\frac{1}{\omega^2}=\frac{LC-RC^2(R+S+h_aRS)}{1+h_aS}.$$

This equation leads to an expression for the wave-length of the assemblage in terms of the constants of the tube and the oscillator.

Again, from the equations for sin φ and cos φ, or directly from the triangle VV′W, we have

$$\frac{I}{J}=h_a\sqrt{\{(gM-L)^2\omega^2+S^2\}},$$

which gives the ratio of the amplitude of the sine current through the tube to that of the sine current in the oscillator.

Numerical Example.

Let $L=10^{-2}$ henry, $C=3\times10^{-10}$ farad,

$R=0{\cdot}1$ ohm, $S=100$ ohms,

$h_a=5\times10^{-5}$ mho.

Then
$$gM\doteqdot10^{-2}+3\times10^{-10}(10+100/5\times10^{-5})$$
$$\doteqdot10^{-2}+6\times10^{-4}.$$

If $g=10$, M must be greater than 10^{-3}, for the maintenance of oscillations.

Again,
$$\frac{1}{\omega^2}\doteqdot\frac{3\times10^{-12}-9\times10^{-19}}{1+5\times10^{-3}}\doteqdot3\times10^{-12}.$$

Also,
$$\frac{I}{J}=5\times10^{-5}\sqrt{\{1{\cdot}2\times10^5+10^4\}}$$
$$\doteqdot5\times10^{-5}\times3\times10^2$$
$$=1/70.$$

Let, now, $R=100$ ohms instead of the value above.

Then
$$gM\doteqdot10^{-2}+10^{-3}$$
$$\frac{1}{\omega^2}\doteqdot\frac{3\times10^{-12}-1{\cdot}8\times10^{-15}}{1+1/200}$$
$$\frac{I}{J}\doteqdot5\times10^{-5}\sqrt{\{3{\cdot}3\times10^5+10^4\}}\doteqdot3\times10^{-3}$$
$$=1/30.$$

These results show that the resistance has very slight effects in circuits where the ratio L/C is large, except as regards the ratio of the currents.

Particular Case.

The diagram and the analysis simplify greatly if the inductance coil L is so well designed that its resistance S is zero. This case is approximated to when an antenna is excited direct by an ionic tube, in which case C is the antenna capacity, L its inductance and R the radiation resistance, &c. The diagram appears in Fig. 5 with all its sides marked, and

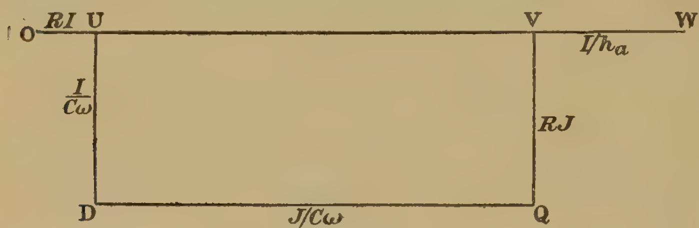


FIG. 5.

needs no explanation. The equations are further easily deduced direct from the diagram or obtained by putting $S=O$ in those above. We obtain

$$gM=L+CR/h_a$$

as the condition assigning the least magnitude to the controlling action, and

$$\frac{1}{\omega^2}=LC-R^2C^2$$

as the frequency equation. The ratio of the currents is

$$\frac{I}{J}=h_a(gM-L)\omega$$
$$=CR\omega.$$

On substituting for ω and squaring, the last equation leads to

$$\frac{J^2}{I^2}=\frac{LC-R^2C^2}{R^2C^2}$$
$$=\frac{L}{R^2C}-1.$$

This points out a limitation not previously mentioned, namely, that

$$L>CR^2.$$

It is noteworthy that in this adjustment of the oscillator it behaves as a mere resistance to the alternating current i; that is to say, it is now non-reactive, and to a steady current it offers no resistance. A formula for the alternating current resistance may be obtained by dividing the length of the line OV in Fig. 5 by the current I, for OV is the amplitude of the P.D. between the terminals of the oscillator and I is the amplitude of the current through it. Calling this quotient Z we may write $\mathrm{OV}=ZI$. But

$$\mathrm{OV}=L\omega J$$

and therefore $$Z=L\omega J/I,$$

whence $$Z=L/RC$$

by aid of a former equation.

The numerical example given in a preceding paragraph gives the following results when S is taken zero and $R=100$ ohms

$$gM=10^{-2}+6\times10^{-4}\doteqdot10^{-2},$$
$$1/\omega^2=3\times10^{-12}-9\times10^{-16}\doteqdot3\times10^{-12},$$
$$I/J\doteqdot1{\cdot}73\times10^{-2}$$
$$Z=\tfrac{1}{3}\,.\,10^6\doteqdot333{,}000 \text{ ohms}.$$

Control by Condenser Coupling.

A very important circuit with excellent oscillating properties is that of which the connections are given in Fig. 6.

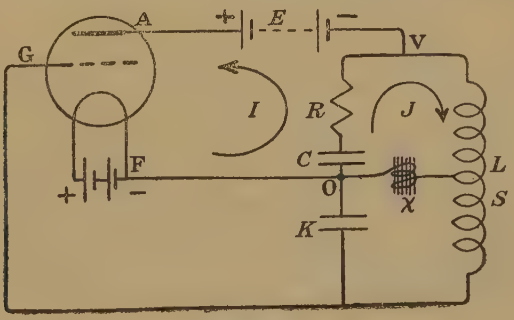


FIG. 6.

We shall assume that when the antenna is made one of the capacitances, that capacitance will be C, and therefore the esistance R may be regarded as antenna resistance. We hall suppose that the inductance L has been designed, as it ught, to be of negligible resistance. A choking coil χ is

connected to the common terminal of the condensers and to a tapping in the inductance so as to provide a path for the steady current from the battery E required to set up the space charge in the tube. The crank diagram is given in Fig. 7, which shows also the notation regarding currents. From it we obtain, as from Fig. 4, the quantitative relations connecting the electrical magnitudes. The angles UOD, DUQ, QVD being equal and the length UV being equal to DQ, give us

$$\frac{J}{I}=\frac{1}{RC\omega}=\frac{I}{XC\omega J}=(Z-R)C\omega.$$

From the first and last we obtain

$$\frac{J^2}{I^2}=\frac{Z-R}{R},$$

and also

$$Z=R+\frac{1}{RC^2\omega^2}$$

The first and second give

$$\frac{1}{R^2C^2\omega}=\frac{1}{XC\omega},$$

which leads to the frequency equation

$$\frac{1}{\omega^2}=(L-R^2C)C_1.$$

In this, C_1 is written for the capacitance of the condensers C and K in series. From the above we have also

$$Z=R+\frac{(L-R^2C)C_1}{RC^2}=\frac{LC_1}{RC^2}+\frac{RC_1}{K}.$$

In the last equation the former of the two terms is easily the more important. Referring to Fig. 6, we see that the fall of potential from G to F must be due to the current J running in the condenser K, and therefore

$$e_g=j/K\omega.$$

In the crank diagram this fall of potential appears as VP and being in phase with OU we see that the oscillation will be constant if this vector VP is great enough. The control voltage e_g when transferred to the plate circuit is of magnitude g times e_g, and has to make up for the fall of potential

in the resistance of a tube—namely, I/h_a—and for the fall in the oscillator. Thus

$$ge_g=(1/h_a+Z)i=(1/h_aZ+1)v.$$

But $$e_g=j/K\omega$$
$$=(Z-R)(C/K)i.$$

Therefore $$g(Z-R)(C/K)=1/h_a+Z.$$

But $$Z-R=\frac{1}{RC^2\omega^2}.$$

Therefore $$g/(RCK\omega^2)=1/h_a+R+1/(RC^2\omega^2),$$
$$g/K=RC\omega^2(1/h_a+R)+1/C,$$
$$C/K \doteqdot RC^2\omega^2/(gh_a)+1/g.$$

This shows that the ratio of the two condensers must exceed the value written on the right-hand side of the last equation.

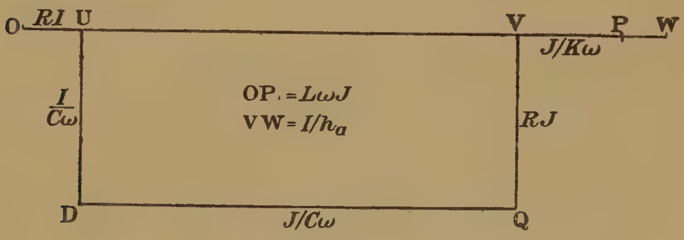


FIG. 7.

The more general case, in which the inductance coil is taken to possess resistance S, is solved by Fig. 8. The grid voltage transferred to the anode circuit must equal the resultant of OV, the fall of potential across the oscillator and VW, the fall of potential across the vacuum, and therefore is represented fully by OW.

The formulæ are obtained from the Figure in a way perfectly analogous to that adopted in Fig. 4. By projection we have

$$XJ=RI\cos\varphi+(I/C\omega)\sin\varphi,$$
$$(R+S)J=(I/C\omega)\cos\varphi-RI\sin\varphi.$$

From the triangle VWV″

$$I\sin\varphi=JSh_a,$$
$$I\cos\varphi=h_a\left(\frac{g+1}{K\omega}-L\omega\right)J.$$

From these we obtain

$$\frac{g+1}{K}=\left\{L+\left(RS+\frac{R+S}{h_a}\right)C\right\}\omega^2$$

as the equation for determining the least magnitude of the control, and

$$\frac{1}{\omega^2}=\frac{L-(R+S+RSh_a)C}{(C+K)/CK+h_aS}$$

for determining the frequency. The ratio of currents is obtained from the first two equations, or directly from the

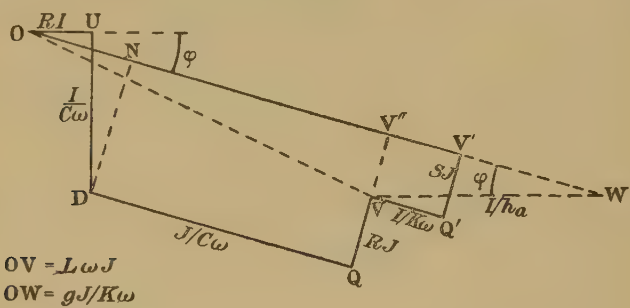


FIG. 8.

NOTE.—The length VQ′ should be marked $J/K\omega$; the length $OV'=L\omega J$.

Figure by expressing OD in terms first of I vectors and then of J vectors; it is

$$\frac{I}{J}=\sqrt{\frac{\left\{L\omega-\left(\frac{1}{C}+\frac{1}{K}\right)\frac{1}{\omega}\right\}^2+(R+S)^2}{(1/C^2\omega^2+R^2)}}.$$

ABSTRACT.

The method of the crank or vector diagrams used commonly in the study of alternating-current circuits is applied in the Paper to the assemblage made up of an oscillator, the thermionic relay maintaining it in oscillation, and the devices linking these two parts. The diagrams then serve as substitutes for the usual treatment of the problem by differential equations and from them may be obtained all the formulæ. They have, besides, the advantage of exhibiting to the eye the phases of the currents and voltages in every part of the circuits. In forming the diagrams the potential drop across the oscillator is calculated by the usual rules of the alternating-current diagram, and added geometrically to the potential drop across the tube. This total is made equal, in magnitude and phase, to the voltage applied at the instant to the grid multiplied by the voltage factor of the relay. In its turn the voltage applied to the relay depends

upon and is obtained from the current running in a portion of the oscillator. The fitting together of these lines gives all the conditions to be satisfied for the maintenance of steady oscillations.

DISCUSSION.

Prof. G. W. O. Howe agreed that you could not really understand the conditions in a circuit unless you were able to put down the currents and voltages in a vector diagram. He had attempted himself to simplify Vallauri's treatment for students, but without as much success as Dr. Eccles. He had usually looked at these circuits from this point of view: The oscillatory circuit apart from the valve has a certain equivalent resistance. To maintain oscillations the equivalent of a negative resistance must be introduced. There are two ways of varying the current in the bulbs; the P.D. on the terminals or on the grid may be varied. In the latter case the current may be increased even if the total P.D. on the bulb is diminished and if this condition can be arrived at we have the equivalent of a negative resistance to the oscillations. You have to arrange a coupling device so that the variations of current in the plate circuit produces suitable variations of potential of the grid. From the characteristic of the plate circuit you can determine the critical value of the mutual inductance between plate and grid circuits to give the equivalent negative resistance necessary for maintenance of steady oscillations.

Prof C. R. Fortescue said the method of approach appears to be a distinct advance, in that the idea of the tube as a generator of an E.M.F., gEg, enables a voltage diagram to be plotted instead of the more usual current diagram. This is an undoubted advance, and simplifies the final adjustment of the diagram to suit the conditions of the tube and circuit. At various times many vector diagrams have been drawn for various oscillatory circuits; and on the whole the results have hardly come up to expectations. There appear to be three reasons for this, viz.: (*a*) In order to draw the diagram at all a very clear insight into the conditions is required. In other words, it is necessary to know the final result before the diagram is completed. In the diagram of Fig. 3 of the Paper it is necessary to know, firstly, that the angle θ is a positive angle, and, secondly, that the effective E.M.F. of the tube is in phase with the voltage applied to the grid. The latter condition is, of course, obvious from the action of the tube, but the former is by no means obvious, and has to be discovered by trial and error. If, for example, the current I is considered to be flowing through the inductance, then θ must be taken as an angle of lag, as will be found if an attempt is made to plot out this diagram. (*b*) The quantities are such that it is impracticable to plot the diagrams to scale. For example, if the numerical values of the first example on page 3 are taken, the ratio of the length of the line OUg of Fig. 3, to the length DQ is of the order of 1 to 500,000. Actually, if Fig. 3 is plotted to scale, it becomes a diagram vertically up and down the board. (*c*) Finally, there is the common experience of the sign difficulty in drawing the diagrams. There are many possibilities of confusion with the ordinary current diagram, but it is possible that with the author's voltage diagram these troubles will be very much reduced. It would appear that the true function of these diagrams, when numerical values are applied or when the diagrams are drawn to scale, is to justify the practical method commonly used of regarding the valve as a power supply. The anode current is in phase with the voltage across the oscillatory circuit. Taking the alternating components only, the product of the anode current and the circuit voltage gives the power supplied to the circuit. The anode current depends upon the anode and grid voltages, *i.e.*,

$$i=h_a e_a+h_g e_g,$$

as given on page 139 of the Paper. To within a small percentage this power is absorbed by the circuit losses. If $R+S$ is the effective resistance

of the oscillatory circuit, then for the oscillations to be maintained or built up, the power supply from the valve must be equal to or greater than the $J^2(R+S)$ loss. This method of dealing with the problem has many practical advantages, and has been in use for some years. Hazeltine has described, in the "Proceedings" of the American Institute of Radio Engineers, the application of the method to various circuits, notably the circuit in which a condenser is connected across the grid inductance h. It is very well known that the author of this Paper has had experience of many other circuits and applications of three electrode relays, and it is to be hoped that he may be able to give to the Society further Papers on this same subject. In particular, a vector treatment of the De Forest ultraudion circuit would be of great interest, as this circuit is one which has presented very great difficulties when any attempts have been made to calculate the condition for instability.

Mr. J. Nicol asked if there was any lag between the voltage applied to the grid and the effect on the current in the valve.

Dr. D. Owen said it was assumed that the action of the valve was an amplification of voltage. Actually what happened was a variation in the resistance of the valve. It seemed rather remarkable that this should be regarded as a voltage effect. He did not like Prof. Howe's idea of a negative resistance, as such a conception had no physical significance. A negative value of dV/dC was by no means the same thing as a negative value of V/C, which would be required before we could talk of a negative resistance Did the author find it satisfactory to treat the quantity h_a as a constant ?

Mr. F. E. Smith thought the Paper of great value. He agreed with Prof. Fortescue that a great amount of interesting work was still to be done in connection with these problems.

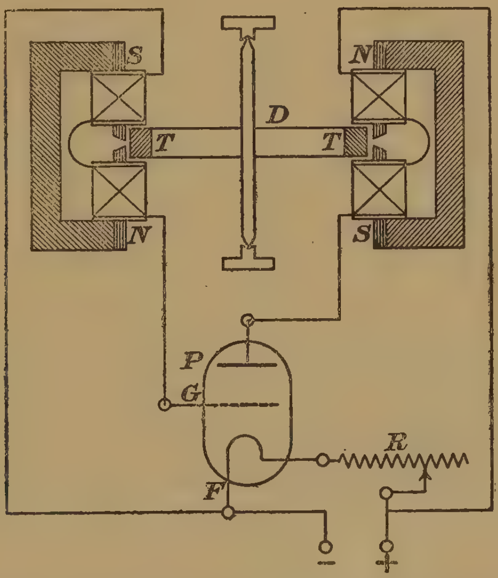
Capt. Turner referred to a sentence occurring in the section dealing with a particular case: "It is noteworthy that . . . no resistance." This seemed to imply that an infinitesimal change in E would produce a finite change in the steady component of the anode current.

Dr. Eccles, in reply, said that he did not think vector diagrams were often used quantitatively. They were mainly used to derive a formula, and the scale of the vectors was immaterial. As regards lag, he was not aware of any definite knowledge on this point, but the fact that nowadays waves of 30,000,000 $\sim$ could be obtained easily showed that the lag must be very small. He thought it was justifiable to assume it zero in slow circuits. The conception of the valve as a voltage amplifier seemed a difficulty ; but if we vary the resistance of one part of a circuit containing a fixed E.M.F. it is clear that the P.D. on the remainder of the circuit will also vary ; so that the varying resistance can be regarded as a source of varying E.M.F. applied to the remainder of the circuit. The quantity h_a was not strictly constant. As regards Capt. Turner's point, he was referring here to a case in which there was no resistance in the inductance circuit.

XIV. *A Small Direct-current Motor Using Thermionic Tubes Instead of Sliding Contacts.* *By* W. H. Eccles, *D.Sc.*, *and* F. W. Jordan, *B.Sc.*

Received January 15, 1919.

In physical laboratories, especially those in which electric waves and oscillations are studied, circumstances sometimes arise in which a wheel or disc has to be spun rapidly under light load and with absolute freedom from the sparking that occurs in the best ordinary direct-current motor. In such cases a motor employing a rotating magnetic field can be used



if alternating current is available, but often alternating current is not at hand. We therefore describe in this Paper a small perfectly sparkless motor that can be run from a direct current supply, such as that used for lighting. Apart from the applications alluded to, this new motor might be used for maintaining gyrostats in rotation, for driving stroboscopes, and so on.

The motor is an application of the three-electrode ionic relay now so well known. In such relays there is a glowing

filament F functioning as cathode, a plate or cylinder P as anode, and an intervening grid G as control electrode. A constant E.M.F. is applied between filament F and anode P and causes a steady stream of electrons to pass from filament to anode across the vacuum. When a control voltage is applied between filament and grid the anode current increases if the grid is made positive relative to the filament, and it diminishes if the grid is made negative. Either terminal of the filament may be taken as the zero of potential, but it is customary to take the negative terminal. For instance, in the small tubes shown with the motor, the anode current may be about 1·5 milliamperes when the grid is at the same potential as the negative terminal of the filament, and 2·5 milliamperes when the grid is at $+5$ volts and 0·4 milliampere when the grid is at -5 volts. The current flowing into the grid in the first case is 150 microamperes and in the last zero. When an alternating voltage is applied to the control electrode an alternating current appears in the anode or repeat circuit superposed upon the steady current that flows in the quiescent state. This alternating current is capable of doing work, and the power thus made available is much greater than that expended in the control circuit—a fact implied in calling the tube a relay.

In the motor here described a number of iron teeth are carried by the rotating part of the motor past an electromagnet connected into the control circuit of the ionic relay, and these teeth generate in the windings of the electromagnet an alternating E.M.F. that is applied to the grid of the tube. The corresponding alternating current in the repeat circuit is sent through a second electromagnet connected in that circuit and also placed near the rotor. Its position relative to the former electromagnet and to the teeth is so adjusted that the alternating current in it tends to accelerate the movement of the rotor. Put briefly, we may say that the passage of an iron tooth in front of the control magnet applies to the grid an E.M.F. that produces, by means of the relay, a current in the second electromagnet in such a direction as to pull forward the tooth just approaching it. In consequence the spin of the rotor increases until frictional and other losses consume the energy liberated from the battery in the anode circuit.

Obviously a motor constructed on these principles may take many different forms. The one exhibited to-day is sketched in the Figure. The rotor is a horizontal ebonite disc 12 cm. in

diameter mounted on a vertical spindle; the electromagnets were polarised magnets from a pair of 4,000 ohm Brown telephone receivers. The iron teeth are twelve in number and fixed at equal distances on the rim of the ebonite disc.

ABSTRACT.

In this motor the rotating part is an ebonite disc with iron teeth on its periphery, and the stationary part comprises two electromagnets with their poles close to two teeth. One electromagnet is connected to the grid of a thermionic relay, the other is included in the plate circuit. When during rotation a tooth passing the grid magnet induces a voltage in its winding the consequent transient increase of current through the other magnet causes this magnet to exert a pull on the tooth approaching it. We thus have a small motor without commutator or spark which may under no-load be driven up to a speed of 4,000 to 6,000 revolutions per minute from the lighting supply.

CORRECTION.

I find that I was quite in error in the Discussion of my "Note on Microphone Hummers" when I suggested that Lord Rayleigh had sometimes called a mere overtone a harmonic. On the contrary, he has always maintained a clear distinction between the two terms.—ALBERT CAMPBELL.

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